

APPROXIMATION DYNAMICS

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ABSTRACT. We describe the approximation of a continuous dynamical system on a p. l. manifold or Cantor set by a tractable system. A system is tractable when it has a finite number of chain components and, with respect to a given full background measure, almost every point is generic for one of a finite number of ergodic invariant measures with non-overlapping supports. The approximations use non-degenerate simplicial dynamical systems for p. l. manifolds and shift-like dynamical systems for Cantor Sets.

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For Roy Adler, who taught us all so much.

1. TRACTABLE DYNAMICAL SYSTEMS AND APPROXIMATION

Our spaces will all be compact metric spaces equipped with a metric d . For a subset A we use \overline{A} and A° for the closure and interior of A , respectively. With $\epsilon \geq 0$ we let $\bar{V}_\epsilon = \{(x, y) : d(x, y) \leq \epsilon\}$.

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A *dynamical system* is a pair (X, f) with f a continuous map on a such a space X . The dynamics is given by iterating the map. A subset A of X is called *f invariant* when $f(A) = A$.

Given $\epsilon > 0$ an ϵ *chain* is a sequence $\{x_0, \dots, x_n\}$ with $n \geq 1$, such that $d(f(x_i), x_{i+1}) \leq \epsilon$ for $0 \leq i < n$. An infinite ϵ chain for a map f is often called an ϵ *pseudo-orbit* of f . The chain relation $\mathcal{C}f \subset X \times X$ consists of those pairs $(x, y) \in X \times X$ such that for every $\epsilon > 0$ there exists an ϵ chain with $x_0 = x$ and $x_n = y$.

In observing a system, for example in a computer simulation, there is usually some positive ϵ level of observational or computational error. The latter implies that in attempting to compute an orbit sequence, we are, in fact, producing an ϵ pseudo-orbit.

A point $x \in X$ is *chain recurrent* if $(x, x) \in \mathcal{C}f$ and two such points x, y are *chain equivalent* if $(x, y), (y, x) \in \mathcal{C}f$. The chain equivalence classes of chain recurrent points are closed, f invariant subsets of X called *chain components* or *f basic sets* (we will use the latter term).

Chain recurrence is the broadest notion of recurrence. If $x = f(x)$ then x is an *equilibrium point* or *fixed point* for f . If $x = f^n(x)$ for some positive integer n then x is a *periodic point*. Let $\omega f(x)$ denote the set of limit points of the orbit sequence $\mathcal{O}f(x) = \{f(x), f^2(x), \dots\}$. If $x \in \omega f(x)$, then x is a recurrent point. For any $x \in X$, the set $\omega f(x)$ is a nonempty, closed, f invariant subset of X which is contained in a single basic set.

The simplest sort of dynamical system has a finite chain recurrent set. In that case, each basic set is a periodic orbit and $\omega f(x)$ is a periodic orbit for every $x \in X$. Beginning from an initial point the orbit exhibits a transient motion towards its limiting periodic orbit. Eventually, we observe what appears to be the periodic motion of the limit orbit.

We will focus upon the more general case when the chain recurrent set decomposes into only finitely many basic sets. For a basic set B we will define the *basin* of B to be the open set $G(B) = \{x : \omega f(x) \subset B\}^\circ$. We will call B a *visible basic set* if $G(B) \neq \emptyset$. When there are only finitely many basic sets then the union of the basins is a dense open subset of X (see Theorem 2.1 below).

In that case, the orbit every point approaches a basic set and most points approach a visible basic set. However, the motion within a basic set may be quite complicated, i.e. chaotic. So to describe such motion we resort to statistics using an invariant measure.

By a *measure* μ on X we mean a Borel probability measure. The *support* $|\mu|$ of the measure is the set of points whose every neighborhood has positive measure. The measure is called *full* when $|\mu| = X$. The

measure is *f* invariant when $\mu(f^{-1}(A)) = \mu(A)$ for every Borel set A , or, equivalently, when $\int (u \circ f) d\mu = \int u d\mu$ for every continuous real-valued function u on X . An invariant measure is *ergodic* if for any Borel set A , $f^{-1}(A) = A$ implies $\mu(A) = 0$ or 1 .

A point x is called *generic* for an invariant measure μ if

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} u(f^i(x)) = \int u d\mu$$

for every continuous real-valued function u . We let $Gen(\mu)$ denote the set of points generic for μ . We do not assume that $x \in |\mu|$. The Birkhoff Pointwise Ergodic Theorem says that if μ is an ergodic measure, then $\mu(|\mu| \cap Gen(\mu)) = 1$. If $x \in |\mu| \cap Gen(\mu)$ then $\omega f(x) = |\mu|$ and so $|\mu|$ is contained in a basic set. In particular, if μ is ergodic then $|\mu|$ is contained in a basic set.

The tractable systems (X, f) that we will consider occur on spaces equipped with some background measure μ_0 which is full but not usually invariant.

Definition 1.1. *Let (X, f) a dynamical system with μ_0 a full measure on X . We will call (X, f) tractable when*

- TRAC 1 *There are only finitely many basic sets.*
- TRAC 2 *There are a finite number of ergodic invariant probability measures μ_1, \dots, μ_k such that $\mu_0(Gen(\mu_i)) > 0$ for $i = 1, \dots, k$ and $\mu_0(\bigcup_{i=1}^k Gen(\mu_i)) = 1$.*
- TRAC 3 *For $i = 1, \dots, k$, if B_i is the basic set containing $|\mu_i|$ then $\{B_1, \dots, B_k\}$ are the visible basic sets.*
- TRAC 4 *It may happen that a visible basic set contains more than one $|\mu_i|$, but if $i \neq j$ then the intersection $|\mu_i| \cap |\mu_j|$ has μ_0 measure zero and so is nowhere dense.*

Tractable systems do exist, e.g. Morse-Smale diffeomorphisms and the Axiom A diffeomorphisms described by Smale in [8]. On the other hand, the generic homeomorphisms on manifolds of positive dimension and on Cantor sets have infinitely many attractors and uncountably many basic sets, see [3].

Rather than look for conditions which will assure that a system is tractable, our approach here will be to approximate an arbitrary system by a tractable ones in a natural way.

Our approximations will be built using finite data. The analogy is with curve sketching. To sketch the graph of a real-valued function on a bounded interval we plot a finite number of points and then connect the dots with line segments. The resulting graph is piecewise linear and

is completely determined by the end-points of each segment, but we use the p.l. function instead of the dots because then our approximation is the same sort of object as that which we wish to study, i.e. a real-valued function on the interval.

The advantage of these approximations is that the dynamics is directly computable from the finite initial data. The disadvantages are two-fold. First, we are operating in the topological category so that our approximating functions g are merely C^0 close to the original function f . The g orbit of a point is a pseudo-orbit of the original function f . Second, our functions g are never invertible and so if f is a homeomorphism and we want to describe the backward as well as forward dynamics, then we require separate approximations for f and f^{-1} . On the other hand, computer simulations usually produce only pseudo-orbits and so we expect that our approximations should pick up the sorts of things observed by a simulation.

In the next two sections we review the dynamics of relations. In addition to the dynamics of a continuous map on a general compact metric space, we are interested in the case when $G \subset K \times K$ with K a finite set. The two situations are related in that associated with such a relation G is the *sample path space*, a closed subset K_G of $K^{\mathbb{Z}_+}$ on which the shift map restricts to a subshift of finite type. A *stochastic cover* Γ of G is a stochastic matrix with $\Gamma_{ji} > 0$ iff $(i, j) \in G$. Such a stochastic cover induces a Markov chain with associated Markov measures on K_G .

We will see in Section 3 that from the stochastic cover we obtain a background measure so that the shift on K_G becomes the paradigmatic example of a tractable system. We also consider the so-called *special two-alphabet model* which is applied in the later sections. Such a model consists of two finite sets K^* and K together with maps $J, \gamma : K^* \rightarrow K$. We then obtain the relations $G = \gamma \circ J^{-1}$ and $G^* = J^{-1} \circ \gamma$ on K and K^* , respectively. The classic example would be a directed graph with K^* the set of edges and K the set of vertices. The maps J and γ associate to an edge its initial and terminal vertex, respectively. When the map J in the two-alphabet model is surjective, we can choose for each $s \in K$ a distribution on $J^{-1}(s) \subset K^*$. From this *distribution data* there naturally arise stochastic covers for the relations G and G^* .

In Section 4, X is a d -dimensional p.l. manifold. We approximate the continuous map f on X by a non-degenerate *simplicial dynamical system*. Let K be a simplicial complex triangulating X and K^* be a subdivision which is *proper*, i.e. no simplex of K^* meets disjoint simplices of K . Let dK and ${}^dK^*$ be the sets of d -dimensional simplices in each complex. Let $J : {}^dK^* \rightarrow {}^dK$ be the map associating to each $s^* \in {}^dK^*$ the unique $s \in {}^dK$ which contains it. A non-degenerate

simplicial dynamical system is a simplicial map $\gamma : K^* \rightarrow K$ such that the dimension of $s = \gamma(s^*) \in K$ is equal to that of s^* for every simplex $s^* \in K^*$. So γ restricts to a map from ${}^d K^*$ to ${}^d K$. Furthermore, γ can be chosen so that the associated p.l. map g on X is uniformly close to f . A simplicial map is determined by its value on the vertices and so is given by finite data. The relation $G^* = J^{-1} \circ \gamma$ on the finite set ${}^d K^*$ induces the subshift $(({}^d K^*)_{G^*}, S)$. There is an almost one-to-one map taking subshift onto (X, g) , thus providing a simple description of the dynamics of g . Furthermore, with a locally Lebesgue measure taken as background measure on X , the system (X, g) is tractable.

Finally, in Section 5 X is a Cantor set $A^{\mathbb{Z}_+}$ with A a finite alphabet, we fix two positive integers n and k . Let K and K^* be the sets of words in the alphabet of length n and $n + k$, respectively. Let $J_n : X \rightarrow K$ and $J_{n+k} : X \rightarrow K^*$ associating to $x \in X$ the initial word of length n or $n + k$. Similarly, $J : K^* \rightarrow K$ maps each $n + k$ word to its initial n word. If $\gamma : K^* \rightarrow K$ is an arbitrary map, the associated *shift-like* map g on X is given by $g(s^*z) = \gamma(s^*)z$ where $s^* = J_{n+k}(x)$ for $x = s^*z$ and so $\gamma(s^*) = J_n(g(x))$. The relation $G^* = J^{-1} \circ \gamma$ on the finite set K^* induces the subshift $((K^*)_{G^*}, S)$. There is a homeomorphism taking the subshift onto (X, g) and so the two systems are conjugate. Furthermore, with the uniform Bernoulli measure taken as background measure on X , the system (X, g) is tractable. For any continuous function f on X and any positive integer n , there exists a positive integer k so that $J_n(f(x))$ depends only on $J_{n+k}(x)$ and so $\gamma = J_n \circ f \circ J_{n+k}^{-1}$ is a map from K^* to K . If g is the shift-like map associated with γ then for all $x \in X$, $J_n(f(x)) = J_n(g(x))$ and so g approximates f . In addition, there is a continuous map Q^f on X which maps f to g . For every x , $J_{n+k}(Q^f(x)) = J_{n+k}(x)$. Since Q^f maps f to g , the g orbit of $Q^f(x)$ shadows the f orbit of x .

2. RELATION DYNAMICS AND TRACTABLE SYSTEMS

One can define many dynamics notions when one merely assumes that f is a closed relation on a space X , i.e. a closed subset of $X \times X$. A (closed) relation $f : X \rightarrow Y$ is just a (closed) subset of $X \times Y$. We let $f(x) = \{y : (x, y) \in f\}$ and $f(A) = \bigcup_{x \in A} f(x)$. For example, $\bar{V}_\epsilon(x)$ is the closed ϵ ball centered at x . The relation f is a map, exactly when $f(x)$ is a singleton for every x . In that case we will use the same symbol for the set and the point therein. For example, the identity map 1_X

on X is the diagonal set $\{(x, x) : x \in X\}$. A map is continuous iff it is closed as a relation.

If $f : X \rightarrow Y, g : Y \rightarrow Z$ are relations then $g \circ f : X \rightarrow Z$ is $\{(x, z) : \text{for some } y \in Y, (x, y) \in f \text{ and } (y, z) \in g\}$. Thus, $g \circ f$ is the projection to $X \times Z$ of $(f \times Z) \cap (X \times g) \subset X \times Y \times Z$. If f is a relation on X , i.e. a subset of $X \times X$, then we define $f^0 = 1_X$, the identity map; $f^1 = f$ and, inductively, $f^{n+1} = f \circ f^n$. Since composition is associative it follows that $f^{n+m} = f^n \circ f^m$ for any non-negative integers n, m . For $f : X \rightarrow Y$, we let $f^{-1} : Y \rightarrow X$ be $\{(y, x) : (x, y) \in f\}$. Clearly, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. For a relation f on X we let $f^{-n} = (f^{-1})^n$. If $A \subset X$ and f is a relation on X , A is called *+invariant* if $f(A) \subset A$ and *invariant* if $f(A) = A$.

A relation f on X is *reflexive* if $1_X \subset f$, *symmetric* if $f^{-1} = f$ and *transitive* if $f \circ f \subset f$.

From a closed relation f on X we construct other relations on X .

- The *orbit relation*: $\mathcal{O}f = \bigcup_{n=1}^{\infty} f^n$.
- The *limit point relation*: ωf is defined by $\omega f(x) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m \geq n} f^m(x)}$.
- The *chain relation*: $\mathcal{C}f = \bigcap_{\epsilon > 0} \mathcal{O}(\bar{V}_\epsilon \circ f)$.

The orbit relation is transitive, and the chain relation is both closed and transitive.

If X is finite, and so is discrete, then $\mathcal{O}f = \mathcal{C}f$.

For a relation f on X we let $|f| = \{x \in X : (x, x) \in f\}$ which is a closed set when f is a closed relation. Extending the language from the case when f is a continuous map, we call the points of $|f|$ the *fixed* points of f , $|\mathcal{O}f|$ the *periodic* points, $|\omega f|$ the *recurrent points* and $|\mathcal{C}f|$ the *chain recurrent* points. On the closed set $|\mathcal{C}f|$ the relation $\mathcal{C}f$ restricts to a reflexive, transitive relation. It can be shown that $\mathcal{C}(f^{-1}) = (\mathcal{C}f)^{-1}$ and so we can omit the parentheses. $\mathcal{C}f \cap \mathcal{C}f^{-1}$ is a closed equivalence relation on $|\mathcal{C}f|$ and the equivalence classes are called the *f basic sets* or *chain components* of f . $\mathcal{C}f$ induces a partial order on the set of basic subsets. When f is a continuous map, each basic set and each limit point set $\omega f(x)$ is f invariant.

We call a closed relation f on X *chain transitive* when $\mathcal{C}f = X \times X$.

A continuous map f on X is *topologically transitive* if pair every pair of nonempty open subsets U, V of X , there exists $i \in \mathbb{Z}_+$ such that $U \cap f^{-i}(V) \neq \emptyset$. For a continuous map f on X we call a point $x \in X$ a *transitive point* when $\mathcal{O}f(x) = X$, or, equivalently, when $\omega f(x) = X$. A continuous map is topologically transitive iff it admits a transitive

point and in that case the set of transitive points is an invariant, dense G_δ subset of X .

If A is a closed subset of X then the *restriction* $f|A$ of a closed relation f on X is the closed relation $f \cap (A \times A)$ on A . If f is a map on X then $f|A$ is a map on A iff A is $^+$ invariant. The subset A is called a chain transitive subset for a map f when the restriction $f|A$ is a chain transitive map. For example, for a map, each limit point set is a chain transitive subset (Proposition 4.14 of [1]) and so is contained in a single basic set.

For a continuous map f on X a closed subset U is called *inward* if $f(U) \subset U^\circ$. A set A is an *attractor* when there is an inward set U such that $\bigcap_{n=0}^\infty f^n(U) = A$. An attractor A is $\mathcal{C}f$ invariant, i.e. $\mathcal{C}f(A) = A$.

For a continuous map f and a closed, $^+$ invariant set A , we will call the open set $G(A) = \{x : \omega f(x) \subset A\}^\circ$ the *basin* of A . For an attractor A , the set $\{x : \omega f(x) \subset A\}$ is itself open and so is the *basin* of the attractor. We will call a basic set B *visible* when it has a non-empty basin.

A basic set is called *terminal* when it is $\mathcal{C}f$ invariant. That is, for a terminal basic set B , $x \in B$ implies $\mathcal{C}f(x) \subset B$. Equivalently, B is a terminal basic set iff it is minimal in the collection of nonempty $\mathcal{C}f$ $^+$ invariant sets. It follows from the usual Zorn's Lemma argument, that any nonempty, closed $\mathcal{C}f$ $^+$ invariant set contains a terminal basic set.

For more detail on these matters see [1].

Theorem 2.1. *Assume that (X, f) is a dynamical system with only finitely many basic sets. Each terminal basic set is an attractor and so is visible. Furthermore, the union of the basins of the basic sets is a dense open subset of X .*

Proof: Let B be a terminal basic set. Number the basic sets $\{B_1, \dots, B_n\}$ so that $B_n = B$. We proceed by induction on n .

Since there are only finitely many basic sets, $B_n \cap |\mathcal{C}f| = |\mathcal{C}f| \setminus (\bigcup_{i < n} B_i) \cap |\mathcal{C}f|$ and so is clopen in $|\mathcal{C}f|$. Since B_n is terminal, $\mathcal{C}f(B_n) = B_n$ and so B_n is an attractor by Theorem 3.3 of [1]. From Proposition 3.9 of [1] it follows that $\{x : \omega f(x) \subset B_n\}$ is the basin $G(B_n)$ and its complement, Y , is the dual repeller. This implies that $\mathcal{C}f^{-1}(Y) = Y$. Let \hat{G} be the interior of Y . Since Y is closed, $G(B_n) \cup \hat{G}$ is a dense open subset of X .

If $\hat{G} = \emptyset$ then the union of the basins is $G(B_n)$ and it is dense in X . In particular, this proves the initial step of the induction.

Assume instead that \hat{G} is non-empty.

By Theorem 3.5 of [1] $\mathcal{C}(f|Y) = (\mathcal{C}f)|Y$ and this implies that the basic sets of $(Y, f|Y)$ are B_1, \dots, B_{n-1} . Let G_i be the $f|Y$ basin of the basic set B_i for $i < n$. Each G_i is open in the relative topology of Y so $G_i \cap \hat{G}$ is open in X . It is clearly contained in (and, in fact, equals) the f basin $G(B_i)$. By induction hypothesis, $\bigcup_{i < n} G_i$ is dense in Y . Since \hat{G} is open, $\bigcup_{i < n} G_i \cap \hat{G}$ is dense in \hat{G} . It follows that $G(B_n) \cup \bigcup_{i < n} G_i \cap \hat{G} \subset \bigcup_{i \leq n} G(B_i)$ is dense in X . This completes the proof of the inductive step.

□

If f_1 and f_2 are relations on X_1 and X_2 respectively, then we say that a continuous map $h : X_1 \rightarrow X_2$ maps f_1 to f_2 if $h \circ f_1 \circ h^{-1} (= (h \times h)(f_1))$ is a subset of f_2 or, equivalently, $h \circ f_1 \subset f_2 \circ h$. If f_1 and f_2 are maps then this implies $h \circ f_1 = f_2 \circ h$ since inclusion implies equality for maps.

Proposition 2.2. *Let (X_1, f_1) and (X_2, f_2) be dynamical systems, so that f_1 and f_2 are continuous maps. Let $h : X_1 \rightarrow X_2$ be a continuous surjection mapping f_1 to f_2 .*

(a) *h maps f_1^{-1} to f_2^{-1} , and for $\mathcal{A} = \mathcal{O}, \omega$, and \mathcal{C} , h maps $\mathcal{A}f_1$ to $\mathcal{A}f_2$.*

(b) *For $\mathcal{A} = \mathcal{O}, \omega$, and \mathcal{C} , $h(|\mathcal{A}f_1|) \subset |\mathcal{A}f_2|$.*

(c) *If B_1 is an f_1 basic set then there exists an f_2 basic set B_2 such that $h(B_1) \subset B_2$.*

(d) *If B_2 is an f_2 basic set then there exists an f_1 basic set B_1 such that $h(B_1) \subset B_2$. If B_2 is a terminal basic set then B_1 can be chosen to be terminal.*

Proof: (a) is an easy exercise and clearly implies (b).

(c) From (b), $h(B_1) \subset |\mathcal{C}f_2|$ and from (a) h maps $\mathcal{C}f_1 \cap \mathcal{C}f_1^{-1}$ to $\mathcal{C}f_2 \cap \mathcal{C}f_2^{-1}$. Hence, the points of $h(B_1)$ lie in a single basic set B_2 .

(d) The basic set B_2 is f_2 invariant and so $h^{-1}(B_2)$ is at least f_1 $^+$ invariant. Since h is surjective there exists x with $h(x) \in B_2$. Since $h^{-1}(B_2)$ is closed and $^+$ invariant $\omega f_1(x) \subset h^{-1}(B_2)$. There exists a basic set $B_1 \supset \omega f_1(x)$. By (c) $h(B_1)$ is contained in some basic set \hat{B}_2 . But $h(B_1)$ meets B_2 and distinct basic sets are disjoint. Hence, $\hat{B}_2 = B_2$.

If B_2 is terminal, and so is $\mathcal{C}f_2$ invariant, then $h^{-1}(B_2)$ is $\mathcal{C}f_1$ $^+$ invariant and nonempty. So it contains a minimal nonempty $\mathcal{C}f_1$ $^+$ invariant set B_1 and this is a terminal basic set.

□

For every point x , the limit point set $\omega f(x)$ is contained in some basic set. Thus, eventually, the motion is close to some basic set. To analyze the behavior within or close to a basic set, we use invariant measures.

The space of signed measures on X is the dual space to the Banach space of continuous, real-valued functions on X . On it we use the weak* topology and in it the set $P(X)$ of probability measures is a compact, convex subset. For $x \in X$ we let δ_x denote the point measure concentrated on x . If $\mu \in P(X)$, then the *support* $|\mu| = \{x : \nu(U) > 0 \text{ for every neighborhood } U \text{ of } x\}$. That is, the closed set $|\mu|$ is the complement of the union of the open sets U of μ measure zero. Since X has a countable base, $X \setminus |\mu|$ is the maximum open set of measure zero. μ is called *full* if $|\mu| = X$, i.e. every non-empty open subset has positive measure.

If $f : X_1 \rightarrow X_2$ is a continuous map, then $\mu \mapsto f_*(\mu)$ is defined by $f_*(\mu)(A) = \mu(f^{-1}(A))$ for all Borel sets $A \subset X_2$ or, equivalently, if $\int u(y)f_*(\mu)(dy) = \int u(f(x))\mu(dx)$ for every continuous $u : X_2 \rightarrow \mathbb{R}$. For example, $f_*(\delta_x) = \delta_{f(x)}$. Thus, $\mu \mapsto f_*\mu$ is a continuous map $f_* : P(X_1) \rightarrow P(X_2)$. If f is a continuous map on X then f_* is a continuous map on $P(X)$ and μ is *invariant* if $f_*\mu = \mu$. Thus, $|f_*| \subset P(X)$ is the set of invariant measures.

For measures μ, ν on X we say that ν is *absolutely continuous* with respect to μ , written $\mu \gg \nu$, if $\mu(A) = 0$ implies $\nu(A) = 0$ for all Borel sets A . The measures are *absolutely equivalent*, written $\mu \approx \nu$, if $\mu \gg \nu$ and $\nu \gg \mu$, i.e. they have the same sets of measure zero.

A measure $\mu \in |f_*|$ is *ergodic* if $f^{-1}(A) = A$ for a Borel set A implies $\mu(A) = 0$ or 1, or, equivalently, if $u : X \rightarrow \mathbb{R}$ is a bounded measurable function then $u \circ f = u$ implies $u(x) = \int u(y)\mu(dy)$ for μ almost every $x \in X$. A point x is called *generic* for μ if for every continuous $u : X \rightarrow \mathbb{R}$, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} u(f^i(x)) = \int u(x)\mu(dx)$, or, equivalently, if μ is the limit in $P(X)$ of the sequence $\{\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}\}$. We write $Gen(\mu)$ for the set of points generic for μ .

The *Birkhoff Pointwise Ergodic Theorem* says that for an ergodic measure μ , μ almost every point of $|\mu|$ is generic for μ , i.e. $Gen(\mu) \cap |\mu|$ has μ measure one.

Notice that we do not assume that $x \in |\mu|$ for $x \in Gen(\mu)$.

Proposition 2.3. *Let μ be an invariant measure for (X, f) .*

- (a) *The support $|\mu|$ is f invariant.*
- (b) *A point $x \in Gen(\mu)$ iff $f(x) \in Gen(\mu)$. In particular, $Gen(\mu)$ is $^+$ invariant.*
- (c) *If $x \in Gen(\mu)$ then $|\mu| \subset \omega f(x)$ and the inclusion can be proper.*

- (d) If $x \in \text{Gen}(\mu) \cap |\mu|$ then $|\mu| = \omega f(x)$ and x is a transitive point for $f|_{|\mu|}$.
- (e) If $\text{Gen}(\mu) \cap |\mu|$ is nonempty, e.g. if μ is ergodic, then $|\mu|$ is a topologically transitive subset of X and so is contained in a basic set.

Proof: (a) Let $x \in X$.

If $x \in |\mu|$ and U is a neighborhood of $f(x)$ then $\mu(U) = \mu(f^{-1}(U)) > 0$ since $f^{-1}(U)$ is a neighborhood of x . So $|\mu|$ is f^{-1} -invariant.

Now suppose $f^{-1}(x) \cap |\mu| = \emptyset$. Each point of $f^{-1}(x)$ has a neighborhood of measure zero. By using the union of a finite subcover, we obtain an open set V of measure zero which contains $f^{-1}(x)$. For example, if $f^{-1}(x) = \emptyset$ then let $V = \emptyset$. There exists an open set U containing x such that $f^{-1}(U) \subset V$. Hence, $\mu(U) = \mu(f^{-1}(U)) = 0$. Hence, $x \notin |\mu|$. Contrapositively, $x \in |\mu|$ implies $f^{-1}(x) \cap |\mu| \neq \emptyset$. Hence, $|\mu|$ is f -invariant.

(b) The sequences $\{\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}\}$ and $\{\frac{1}{n} \sum_{i=1}^n \delta_{f^i(x)}\}$ are asymptotic.

(c) If $y \notin \omega f(x)$, then choose $u : X \rightarrow [0, 1]$ continuous with $u(y) = 1$ and the closure of $\{u > 0\}$ disjoint from $\omega f(x)$. It follows that $u(f^i(x)) > 0$ for only finitely many i and so $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} u(f^i(x)) = 0$. If $x \in \text{Gen}(\mu)$ then $\int u d\mu = 0$. Hence, $u = 0$ on the support of μ . In particular, $y \notin |\mu|$.

If (X, f) is *uniquely ergodic*, i.e. it admits a unique invariant measure μ , then the sequence $\{\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}\}$ converges to μ for every $x \in X$, i.e. $\text{Gen}(\mu) = X$. In [1] Theorem 9.2 a non-trivial, topologically transitive system (X, f) is described with a fixed point e such that δ_e is the unique invariant measure. If x is a transitive point, then $x \in \text{Gen}(\delta_e)$ with $\{e\} = |\delta_e|$, but $\omega f(x) = X$.

(d) If $x \in |\mu|$ then $\omega f(x) \subset |\mu|$ because the support is closed and invariant. Hence, if $x \in \text{Gen}(\mu) \cap |\mu|$ then $|\mu| = \omega f(x)$ and so x is a transitive point for $f|_{|\mu|}$.

(e) Obvious from (d).

□

If $f^n(x) = x$, i.e. x is a periodic point, then $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$ is the uniform invariant measure concentrated on the periodic orbit. It is an ergodic measure. If an ergodic measure is not thus concentrated on a periodic orbit, then it is non-atomic, i.e. countable sets have measure zero.

We will assume that there is a background measure μ_0 on X with μ_0 full, so that the empty set is the only open subset of measure zero.

We also assume that the measure is complete, i.e. any subset of a set of measure zero is measurable with measure zero.

Definition 2.4. Let (X, f) a dynamical system equipped with a full background measure μ_0 on X . We will call (X, f) tractable when

- TRAC 1 *There are only finitely many basic sets.*
- TRAC 2 *There are a finite number of ergodic invariant probability measures μ_1, \dots, μ_k so that every point in the complement of some set of μ_0 measure zero is generic for some (necessarily unique) μ_i with $1 \leq i \leq k$.*
- TRAC 3 *Each $|\mu_i|$ for $1 \leq i \leq k$ is contained in a visible basic set, i.e. a basic set with a nonempty basin.*
- TRAC 4 *If $i \neq j$ then the intersection $|\mu_i| \cap |\mu_j|$ has μ_0 measure zero and so is nowhere dense.*

We can discard any measures μ_i with $\mu_0(\text{Gen}(\mu_i)) = 0$ and so may assume $\mu_0(\text{Gen}(\mu_i)) > 0$ for all i .

The converse condition of TRAC 3 follows automatically from TRAC 2.

Lemma 2.5. *Let (X, f) be a dynamical system with a full background measure μ_0 . Assume that (X, f) satisfies TRAC 1 and TRAC 2. If B is a visible basic set then there exists i such that $|\mu_i| \subset B$. In fact, $\bigcup \{\text{Gen}(\mu_i) : |\mu_i| \subset B\}$ intersects the basin $G(B)$ in a dense subset. In particular, if B is a terminal basic set then there exists i such that $|\mu_i| \subset B$.*

Proof: By definition the basin $G(B)$ is a nonempty open subset of X . Let U be a nonempty open subset of $G(B)$. Since μ_0 is full, U has positive μ_0 measure. By TRAC 2, $\bigcup_{i=1}^k \text{Gen}(\mu_i)$ has μ_0 measure one and so meets any set of positive measure. Hence, there exists $x \in U \cap \text{Gen}(\mu_i)$ for some i . Since $x \in G(B)$, $|\mu_i| \subset \omega f(x) \subset B$ by Proposition 2.3.

If B is terminal, then TRAC 1 and Theorem 2.1 imply that B is an attractor and so is visible.

□

Call a continuous map $h : X_1 \rightarrow X_2$ *almost open* if $h(U) \subset X_2$ has nonempty interior whenever $U \subset X_1$ does. Call it *almost one-to-one* if Inj_h is dense in X_1 where $\text{Inj}_h = \{x \in X_1 : h^{-1}(h(x)) = \{x\}\}$. An almost one-to-one surjection is almost open. In general, if U is a neighborhood of a point $x \in \text{Inj}_h$ then $h(U)$ is a neighborhood of $h(x)$ in the image $h(X_1)$. For a full measure μ_0 on X_1 , h is called *μ_0 almost*

one-to-one if $\mu_0(\text{Inj}_h) = 1$. Since μ_0 is full, μ_0 almost one-to-one implies almost one-to-one.

Lemma 2.6. *Let $h : X_1 \rightarrow X_2$ be an almost one-to-one map and let A_1, A_2 are closed subsets of X_1 . Let μ_0 be a full measure on X_1 .*

(a) *If $A_1 \cap A_2$ is nowhere dense, then $h(A_1) \cap h(A_2)$ is a nowhere dense subset of X_2 .*

(b) *If $h : X_1 \rightarrow X_2$ is μ_0 almost one-to-one and $\mu_0(A_1 \cap A_2) = 0$, then $h_*\mu_0(h(A_1) \cap h(A_2)) = 0$.*

Proof: (a) Suppose that V is an open subset of $h(A_1) \cap h(A_2)$. Because h is almost one-to-one, $h^{-1}(V) \cap \text{Inj}_h$ is dense in the open set $h^{-1}(V)$. By definition of Inj_h , $h^{-1}(V) \cap \text{Inj}_h$ is a subset of the closed set $A_1 \cap A_2$. Hence, $h^{-1}(V) \subset A_1 \cap A_2$. This intersection is nowhere dense and so $h^{-1}(V) = \emptyset$. Since $V \subset h(X_1)$, $V = h(h^{-1}(V)) = \emptyset$.

(b) If $h(A_1) \cap h(A_2)$ had $h_*\mu_0$ positive measure then $h^{-1}(h(A_1) \cap h(A_2))$ would have positive μ_0 measure. Since Inj_h has μ_0 measure one, $\text{Inj}_h \cap h^{-1}(h(A_1) \cap h(A_2))$ has positive measure. But $\text{Inj}_h \cap h^{-1}(h(A_1) \cap h(A_2)) \subset A_1 \cap A_2$ which has measure zero by assumption.

□

Theorem 2.7. *Let (X_1, f_1) and (X_2, f_2) be dynamical systems and let $h : X_1 \rightarrow X_2$ be a continuous surjection mapping f_1 to f_2 . If (X_1, f_1) satisfies TRAC 1 and TRAC 2 with respect to a background measure μ_0 then (X_2, f_2) satisfies TRAC 1 and TRAC 2 with background measure $h_*\mu_0$. If, in addition, h is almost open and (X_1, f_1) satisfies TRAC 3, then (X_2, f_2) satisfies TRAC 3. If, in addition, h is μ_0 almost one-to-one and (X_1, f_1) satisfies TRAC 4 then (X_2, f_2) satisfies TRAC 4.*

Proof: Observe that $h_*\mu_0$ is full when μ_0 is, because h is surjective.

By Proposition 2.2 (d) every f_2 basic set contains the image of an f_1 basic set. since distinct f_2 basic sets are disjoint, it follows that the number of f_2 basic sets is less than or equal to the number of f_1 basic sets.

If $f_2^{-1}(A_2) = A_2$ then $f_1^{-1}h^{-1}(A_2) = h^{-1}f_2^{-1}(A_2) = h^{-1}(A_2)$. It follows that if μ_i is ergodic then $h_*\mu_i$ is ergodic.

If $x \in \text{Gen}(\mu_i)$ then $h(x) \in \text{Gen}(h_*\mu_i)$ by continuity of the map h_* . Hence, if $A_1 = \bigcup_i \text{Gen}(\mu_i)$, then $h(A_1) \subset \bigcup_i \text{Gen}(h_*\mu_i)$. By TRAC 2 for f_1 , $\mu_0(X_1 \setminus A_1) = 0$. Since

$$h^{-1}(X_2 \setminus h(A_1)) = X_1 \setminus h^{-1}(h(A_1)) \subset X_1 \setminus A_1,$$

it follows that $h_*\mu_0(X_2 \setminus h(A_1)) = 0$.

Now assume that h is almost open. If B_1 is a visible f_1 basic set and $h(B_1) \subset B_2$ then the interior of $h(G(B_1))$ is nonempty and is contained in $G(B_2)$. Hence, B_2 is visible. So if $|\mu_i|$ is contained in the visible f_1 basic set B_1 then $|h_*\mu_i|$ is contained in the visible f_2 basic set B_2 .

Finally, assume that h is μ_0 almost one-to-one. If $|\mu_i| \cap |\mu_j|$ has μ_0 measure zero then by Lemma 2.6 (b) $h(|\mu_i|) \cap h(|\mu_j|) = |h_*\mu_i| \cap |h_*\mu_j|$ has $h_*\mu_0$ measure zero. Hence, TRAC 4 for f_1 implies TRAC 4 for f_2 .

□

Condition TRAC 1 is clearly invariant via a conjugacy homeomorphism h , but the others are not except with the change in background measure from μ_0 to $h_*\mu_0$. In general, replacing μ_0 by an absolutely equivalent background measure preserves tractability. So the natural conjugacies for this situation are homeomorphisms h such that $h_*\mu_0$ is absolutely equivalent to μ_0 .

Example: Suppose X is a finite dimensional, connected p.l. manifold without boundary. By the Oxtoby-Ulam Theorem, see [7] and [4], any two full, non-atomic measures on X are homeomorphically equivalent. Now suppose that (X, f) is a tractable system with respect to such an Oxtoby-Ulam measure μ_0 . Suppose that μ_{k+1} is a non-atomic ergodic measure not included among μ_i for $i = 1, \dots, k$. Let $\tilde{\mu}_0 = \frac{1}{2}\mu_0 + \frac{1}{2}\mu_{k+1}$. This is an Oxtoby-Ulam measure and so there exists a homeomorphism \tilde{h} on X such that $\tilde{h}_*(\tilde{\mu}_0) = \mu_0$. Let $\tilde{f} = \tilde{h} \circ f \circ \tilde{h}^{-1}$ so that \tilde{h} maps f to \tilde{f} . Clearly, \tilde{f} satisfies TRAC 2 with background measure μ_0 and list of ergodic measures $\tilde{h}_*(\mu_1), \dots, \tilde{h}_*(\mu_k), \tilde{h}_*(\mu_{k+1})$. If $|\mu_{k+1}|$ is contained in a non-visible basic set, then (X, \tilde{f}) does not satisfy TRAC 3. On the other hand, if $|\mu_{k+1}| = |\mu_i|$ for some $1 \leq i \leq k$ then TRAC 4 fails. Finally, if f admits a sequence of distinct non-atomic ergodic measures $\{\nu_j : j \in \mathbb{N}\}$ then $\hat{\mu}_0 = \frac{1}{2}\mu_0 + \sum_{j=1}^{\infty} \frac{1}{2^{j+1}}\nu_j$ is an Oxtoby-Ulam measure and there exists a homeomorphism \hat{h} mapping $\hat{\mu}_0$ to μ_0 . Let $\hat{f} = \hat{h} \circ f \circ \hat{h}^{-1}$. TRAC 2 does not hold for (X, \hat{f}) .

□

3. SUBSHIFTS FROM RELATIONS AND THE TWO ALPHABET MODEL

For K a finite set, we let $K^{\mathbb{Z}^+}$ be the compact metric space of sequences $\mathbf{s} = (s_0, s_1, \dots)$ in K . On it the shift map S is defined by

$S(\mathbf{s})_i = s_{i+1}$. We define the metric d by

$$(3.1) \quad d(\mathbf{s}, \mathbf{t}) = \inf\{1/(k+1) : s_i = t_i \text{ for all } i < k\}.$$

So if $s_0 \neq t_0$ then $d(\mathbf{s}, \mathbf{t}) = 1$.

The pair $(K^{\mathbb{Z}_+}, S)$ is called the *full shift* with alphabet K .

If G is a relation on a finite set K we define the associated *sample path space* K_G as the closed, invariant subset

$$(3.2) \quad K_G = \{\mathbf{s} \in K^{\mathbb{Z}_+} : (s_i, s_{i+1}) \in G \text{ for all } i \in \mathbb{Z}_+\}$$

The pair (K_G, S) is a subshift of finite type. Let $p_i : K_G \rightarrow K$ denote the projection to the i^{th} coordinate, i.e. $p_i(\mathbf{s}) = s_i$. Each p_i maps the shift map S to the relation G .

Notice that for $\mathbf{s} \in K_G$ every s_i is in the *domain* $\text{Dom}(G) = G^{-1}(K)$. It will be convenient to assume $K = \text{Dom}(G)$. Otherwise, we replace G on K by its restriction to the intersection of the non-increasing sequence $\{G^{-n}(K)\}$.

If G_1 and G_2 are relations on finite sets K_1 and K_2 , and $g : K_1 \rightarrow K_2$ maps G_1 to G_2 , then the product of copies of g restricts to a continuous map $g : K_1 \times_{G_1} \rightarrow K_2 \times_{G_2}$ which maps the shift to the shift.

If A is a subset of K , then, using the restriction of G to A , i.e. the relation $G \cap (A \times A)$ on A , we obtain $A_G = (K_G) \cap (A^{\mathbb{Z}_+})$. The set A_G is a closed, $+$ invariant subset of K_G . We will call $a_0 \dots a_n \in K^{n+1}$ an A_G word if $(a_{i-1}, a_i) \in G \cap (A \times A)$ for $i = 1, \dots, n$.

For any K_G word $a_0 \dots a_n$ we define the clopen *cylinder set* $\langle a_0 \dots a_n \rangle = \{\mathbf{s} \in K_G : s_i = a_i \text{ for } i = 0, \dots, n\}$.

Since $\mathcal{O}G = \mathcal{C}G$, a G basic set B is an $\mathcal{O}G \cap \mathcal{O}G^{-1}$ equivalence class in the set $|\mathcal{O}G|$ of G periodic points. The basic set B is terminal when $\mathcal{O}G(B) \subset B$.

Proposition 3.1. *Let G be a relation on a finite set K with $K = \text{Dom}(G)$.*

- (a) *If $\mathbf{s} \in K_G$ then there exists a G basic set B and $k \in \mathbb{Z}_+$ such that for all $i \geq k$, $s_i \in B$ and $S^i(\mathbf{s}) \in B_G$. The basic set B is called the *endset* of \mathbf{s} , denoted $\text{End}(\mathbf{s})$. With $B = \text{End}(\mathbf{s})$, $\omega S(\mathbf{s}) \subset B_G$.*
- (b) *If $B \subset K$ is a G basic set, then $B_G \subset K_G$ is an S basic set and the restriction $S|_{B_G}$ is topologically transitive. Furthermore, $B \mapsto B_G$ is a bijection from the set of G basic sets in K to the set of S basic sets in K_G .*
- (c) *If $A \subset K$, then $\mathcal{O}G(A) \subset A$ iff $\mathbf{s} \in A_G \Leftrightarrow s_0 \in A$ for $\mathbf{s} \in K_G$. These conditions imply that $\mathcal{C}S(A_G) \subset A_G$, A_G is a clopen subset of K_G , and $\{\mathbf{s} : \text{End}(\mathbf{s}) \subset A\}$ is an open subset of K_G .*

- (d) For every $s \in K$ there exists $\mathbf{s} \in K_G$ with $s_0 = s$ and $\text{End}(\mathbf{s})$ a terminal G basic set.
- (e) For B a G basic set, the following are equivalent:
 - (i) B is a terminal G basic set.
 - (ii) B_G is a terminal S basic set.
 - (iii) B_G is a visible S basic set.
 - (iv) B_G is a clopen subset of K_G .
 - (v) B_G is an attractor for S .
 - (vi) $\{\mathbf{s} : \text{End}(\mathbf{s}) = B\}$ is an open subset of K_G .
- (f) The set $\{\mathbf{s} : \text{End}(\mathbf{s}) \text{ is a terminal } G \text{ basic set}\}$ is a dense open subset of K_G .

Proof: Since $\text{Dom}(G) = K$, any $s \in K$ can be extended to $\mathbf{s} \in K_G$ with $s = s_0$. If $s \in B$ then we can choose $\mathbf{s} \in B_G$. Hence, $p_0(B_G) = B$.

(a) Since K is finite, there exists $t \in K$ such that $s_i = t$ for infinitely many $i \in \mathbb{Z}_+$. Let k be the smallest i such that $s_i = t$. It follows that for all $j \geq k$ s_j is $\mathcal{O}G \cap \mathcal{O}G^{-1}$ equivalent to t and so all lie in a single equivalence class B . It follows that $S^j(\mathbf{s}) \in K_G \cap B^{\mathbb{Z}_+} = B_G$ for all $j \geq k$. Since B_G is closed and S invariant it follows that $\omega S(\mathbf{s}) \subset B_G$.

(b) If $a_0 \dots a_n$ and $b_0 \dots b_m$ are finite B_G words then there exists a finite B_G word $c_0 \dots c_p$ with $c_0 = a_n$ and $c_p = b_0$. So

$$a_0 \dots a_n c_1 \dots c_{p-1} b_0 \dots b_m$$

is a word in B_G which can be extended to an element \mathbf{s} of B_G . We have $\mathbf{s} \in \langle a_0 \dots a_n \rangle$ and $S^{n+p}(\mathbf{s}) \in \langle b_0 \dots b_m \rangle$. Hence, the restriction of S to B_G is topologically transitive.

It follows that B_G is contained in an S basic set Q . Each projection p_i maps S to G and so maps Q into a G basic set which must be B since Q contains B_G . Thus, $Q \subset K_G \cap B^{\mathbb{Z}_+} = B_G$. That is, $Q = B_G$.

Conversely, if Q is an S basic set and $\mathbf{s} \in Q$, let $B = \text{End}(\mathbf{s})$. Suppose $s_j \in B$ for all $j \geq k$ and so $S^j(\mathbf{s}) \in B_G$ for $j \geq k$. On the other hand, $S^j(\mathbf{s}) \in Q$ for all j . Hence, Q and the basic set B_G intersect and so must be equal.

Since p_0 maps B_G onto B , the map $B \mapsto B_G$ has inverse $Q \mapsto p_0(Q)$.

(c) Since any $s \in K$ can be extended to $\mathbf{s} \in K_G$ with $s = s_0$, it easily follows that $A \supset \mathcal{O}G(A)$ iff $\mathbf{s} \in A_G$ whenever $s_0 \in A$. Now assume these conditions hold.

Assume $\mathbf{s} \in A_G$ and $\mathbf{t} \in \mathcal{CS}(A_G)$. There is a finite chain $\mathbf{z}^0, \dots, \mathbf{z}^n$ in K_G with $\mathbf{z}^0 = \mathbf{s}$, $\mathbf{z}^n = \mathbf{t}$ and $z_1^{i-1} = z_0^i$ for $i = 1, \dots, n$. It follows that the sequence $\mathbf{z} = s_0, z_0^1, z_0^2, \dots, z_0^{n-1}, t_0, t_1, \dots$ lies in K_G with $s_0 \in A$. Hence, it is in A_G and so $\mathbf{t} = S^n(\mathbf{z}) \in A_G$.

Since $A_G = \bigcup_{s \in A} \{\langle s \rangle\}$ it follows that A_G is clopen in K_G .

$End(\mathbf{s}) \subset A_G$ iff $\mathbf{s} \in \bigcup_{n=0}^{\infty} S^{-n}(A_G)$. Hence $End(\mathbf{s}) \subset A_G$ is an open condition.

Lemma 3.2. *Assume s_0, \dots, s_n is a K_G word with s_0 in a basic set B and $s_n \notin B$. The set $A = \mathcal{O}G(a_n)$ satisfies $\mathcal{O}G(A) \subset A$ and $A \cap B = \emptyset$.*

Proof: Transitivity of $\mathcal{O}G$ implies $\mathcal{O}G(A) \subset A$. If $s \in A$, then since $(s_0, s_n), (s_n, s) \in \mathcal{O}G$, $s \in B$ would imply $s_n \in B$.

□

(d) Since $Dom(G) = K$, we can extend s to an word with length greater than the cardinality of K and so with repeats. Thus, we can begin with $s_0 = s$ and enter some basic set B_1 . If B_1 is not terminal then we can continue the K_G word so as to eventually exit B_1 . Then, continuing long enough we enter a basic set B_2 and again we can exit if B_2 is not terminal. The process ends when we arrive at terminal basic set. Once we enter in a terminal basic set, we remain in it. This must happen because Lemma 3.2 implies there are no repeats among the basic sets B_1, B_2, \dots . Any continuation of the sequence to $\mathbf{s} \in K_G$ has $s_0 = s$ and $End(\mathbf{s})$ terminal.

(e) From (b) B_G is an S basic set and so from (c) we have (i) \Rightarrow (ii), (iv), and (vi). From (b) there are only finitely many S basic sets and so by Theorem 2.1 a terminal basic set is visible, i.e. (ii) \Rightarrow (iii). Because B_G is a basic set, it is S invariant. A clopen $^+$ invariant set is inward and a clopen invariant set is an attractor. Hence, (iv) \Rightarrow (v).

Thus, (i) implies (ii), (iii), (iv), (v) and (vi).

Now assume (i) is false. By (d), for any $t \in B$ there exists $\mathbf{t} \in K_G$ with $t_0 = t$ and with $End(\mathbf{t})$ terminal and so not equal to B .

Now let $\mathbf{s} \in K_G$ with $End(\mathbf{s}) = B$, e.g. $\mathbf{s} \in B_G$. So there exists N such that $s_n \in B$ for all $n \geq N$. Let n be arbitrarily large with $n \geq N$. By the above remarks, there exists $\mathbf{t} \in K_G$ with $s_n = t_0$ and $End(\mathbf{t})$ a terminal basic set \hat{B} . Concatenating we obtain $\mathbf{z} \in K_G$ with $z_i = s_i$ for $i \leq n$ and with $S^{n+1}(\mathbf{z}) = \mathbf{t}$. Thus, $End(\mathbf{z}) = End(\mathbf{t}) = \hat{B}$ is terminal and not equal to B .

It follows that $\{\mathbf{s} : End(\mathbf{s}) = B\} \supset B_G$ has empty interior. The basic set B_G is not visible and is not open nor is terminal, i.e. not(vi), not(iii), not(iv) and not(ii). Since $\{\mathbf{s} : \omega S(\mathbf{s}) \subset B_G\} = \{\mathbf{s} : End(\mathbf{s}) = B\}$ is not open, B_G is not an attractor, i.e. not(v).

Thus, not(i) implies not(ii), not(iii), not(iv), not(v) and not(vi).

(f) By (b) and (e) the set of visible S basic sets is $\{B_G : B \text{ a terminal } G \text{ basic set}\}$ and for a terminal basic set B the basin $G(B_G) = \{\mathbf{s} :$

$End(s) = B\}$. The density of the union of the basins then follows from Theorem 2.1.

□

For a relation $F : I \rightarrow J$ with I and J finite sets and $Dom(F) = I$, a *stochastic cover* $\Gamma(F)$ is a $J \times I$ matrix such that

$$(3.3) \quad \begin{aligned} \Gamma(F)_{ji} &> 0 \iff (i, j) \in F, \\ \text{and } \sum_{j \in J} \Gamma(F)_{ji} &= 1 \text{ for all } i \in I. \end{aligned}$$

The reversal of order is so that if $G : J \rightarrow K$ is another similar relation with stochastic cover $\Gamma(G)$ then the matrix product $\Gamma(G) \cdot \Gamma(F)$ is a stochastic cover of $G \circ F : I \rightarrow K$.

For the relation G on a finite set K , with $Dom(G) = K$, a stochastic cover $\Gamma(G)$ induces a Markov process with $\Gamma(G)_{s_2 s_1}$ the probability of moving from s_1 to s_2 in a single time step. With $\Gamma(G)^n$ the n -fold product, $\Gamma(G)^n_{s_2 s_1}$ is the probability that the process beginning at s_1 , hits s_2 at the n^{th} step (not necessarily for the first time).

An element K is called *transient* if it is not a member of some terminal G basic set in K . We will let $tran \subset K$ denote the subset of transient elements. If $s \in K_G$ with s_i in a terminal basic set B then $s_j \in B$ for all $j > i$. Contrapositively, if s_j is transient then s_i is transient for all $i < j$. For any positive integer n and $s \in K$, define

$$(3.4) \quad (\Gamma(G)^n)_{tran \ s} = \Sigma\{(\Gamma(G)^n)_{s_1 s} : s_1 \in tran\}.$$

Lemma 3.3. *There exists $0 < \rho < 1$ and a positive integer n such that $(\Gamma(G)^{nk})_{tran \ s} \leq \rho^k$ for all $s \in K$ and positive integers k .*

Proof: By Proposition 3.1 (d) for each s there is an element of K_G which begins at s and eventually is in some terminal G basic set. So by choosing n large enough, for every $s \in K$ there exists a K_G word s_0, \dots, s_n with $s_0 = s$ and s_n in a terminal basic set. This implies that each $(\Gamma(G)^n)_{tran \ s} < 1$. Let ρ be the maximum for all $s \in K$.

Observe that if s_1 lies in some terminal basic set then $(\Gamma(G)^i)_{tran \ s_1} = 0$ for every positive integer i . Hence,

$$(\Gamma(G)^{n(k+1)})_{tran \ s} = \Sigma_{s_1 \in tran} (\Gamma(G)^n)_{tran \ s_1} (\Gamma(G)^{nk})_{s_1 s} \leq \rho (\Gamma(G)^{nk})_{tran \ s}$$

and so the result for all k follows by induction.

□

A *distribution* v on K is a function $v : K \rightarrow [0, 1]$ with $v(s) \geq 0$ for all $s \in K$ and $\Sigma_{s \in K} v(s) = 1$. The *support* of v is $\{s : v(s) > 0\}$. We call v a *positive distribution* when the support is all of K , i.e. $v(s) > 0$ for all $s \in K$. If $A \subset K$ we write $v(A) = \Sigma\{v(s) : s \in A\}$.

We call v a *stationary distribution* for $\Gamma(G)$ when $\Gamma(G) \cdot v = v$, i.e. $\sum_{s_1 \in K} \Gamma(G)_{s_2 s_1} v(s_1) = v(s_2)$.

Proposition 3.4. *Let $\Gamma(G)$ be a stochastic cover of a relation G on a finite set K .*

(a) *If v is a stationary distribution for $\Gamma(G)$ then $v(s) = 0$ for all $s \in \text{tran}$.*

(b) *If B is a terminal basic set, then there is a unique stationary distribution v_B for $\Gamma(G)$ with support equal to B .*

(c) *If v is a stationary distribution then $v = \sum \{v(B)v_B : B \text{ a terminal } G \text{ basic set}\}$.*

Proof: (a) Since $v = \Gamma(G)^{nk} \cdot v$, Lemma 3.3 implies that for every positive integer k , $v(\text{tran}) = \sum_s (\Gamma(G)^{nk})_{\text{tran } s} v(s) \leq \rho^k$. Hence, $v(\text{tran}) = 0$.

(b) The restriction $\Gamma(G)_{s_1 s_2}$ with $s_1, s_2 \in B$ is a an irreducible stochastic matrix when B is a terminal basic set. By the Frobenius theory of non-negative matrices, 1 is the dominant eigenvalue, with one-dimensional eigenspaces, generated by positive eigenvectors. Normalizing the right eigenvector we obtain the unique stationary distribution with support equal to B , see, e.g. Appendix 2 of [6].

(c) This is clear from (a), because the restriction of $\Gamma(G)$ to the union of the terminal basic sets is a block-diagonal stochastic matrix.

□

For the following we refer to [9] Chapter 1.

Proposition 3.5. *Let $\Gamma(G)$ be a stochastic cover of a relation G on a finite set K .*

(a) *For each $s \in K$ there is a measure on $K^{\mathbb{Z}^+}$ with support $\langle s \rangle = \{\mathbf{s} \in K_G : s_0 = s\}$, uniquely defined so that on the cylinder set $\langle s_0 \dots s_n \rangle$*

$$(3.5) \quad \mu_s \langle s_0 \dots s_n \rangle = \Gamma(G)_{s_n s_{n-1}} \dots \Gamma(G)_{s_2 s_1} \Gamma(G)_{s_1 s}$$

if $s = s_0$ and $= 0$ otherwise.

(b) *If B is a terminal basic set with associated stationary distribution v_B then $\mu_B = \sum_{s \in B} v_B(s) \mu_s$ is an ergodic invariant measure for the shift S with support B_G .*

□

For any distribution v on K which is positive, i.e. $v(s) > 0$ for all $s \in K$, we obtain a measure $\mu_v = \sum_{s \in K} v(s) \mu_s$ with full support on K_G . If there are any transient elements of K , then v is not stationary by

Proposition 3.4 (a) and so the measure is not invariant. However, such measures will play the role of the required background measure. In particular, the notion of μ_v measure zero is independent of the choice of positive distribution v since $\mu_v(A) = 0$ iff $\mu_s(A) = 0$ for all s .

Definition 3.6. Let $\Gamma(G)$ be a stochastic cover of a relation G on a finite set K . A point $\mathbf{s} \in K_G$ is $\Gamma(G)$ -generic if $B = \text{End}(\mathbf{s})$ is a terminal G basic set and $\mathbf{s} \in \text{Gen}(\mu_B)$.

Lemma 3.7. If B is a terminal G basic set and $\mathbf{s} \in \text{Gen}(\mu_B)$ then $\text{End}(\mathbf{s}) = B$.

Proof: Let $\hat{B} = \text{End}(\mathbf{s})$ so that $\omega S(\mathbf{s}) \subset \hat{B}_G$. If $\mathbf{s} \in \text{Gen}(\mu_B)$ then by Proposition 2.3(c) $B_G = |\mu_B| \subset \omega S(\mathbf{s}) \subset \hat{B}_G$. As distinct basic sets are disjoint it follows that $B_G = \hat{B}_G$ and so, by Proposition 3.1(b), $B = \hat{B} = \text{End}(\mathbf{s})$.
□

Theorem 3.8. Let $\Gamma(G)$ be a stochastic cover of a relation G on a finite set K and let v be a positive distribution on K . With respect to μ_v the set of points which are not $\Gamma(G)$ -generic has measure zero.

Proof: We will call a point of K_G non-generic if it is not $\Gamma(G)$ -generic.

A set has μ_v measure zero iff it has μ_s measure zero for all $s \in K$. Fix such an s .

A point \mathbf{s} is certainly non-generic if $\text{End}(\mathbf{s})$ is not terminal, or, equivalently, if $s_i \in \text{tran}$ for all i . Given $s \in K$, Lemma 3.3 easily implies that $\{\mathbf{s} : s_0 = s \text{ and } s_i \in \text{tran} \text{ for all } i\}$ has μ_s measure zero. Note that the set is empty if s is not itself transient. It follows that the set of \mathbf{s} such that $\text{End}(\mathbf{s})$ is not terminal has μ_s measure zero.

If B is a terminal basic set then the Birkhoff Ergodic Theorem says that $B_G \cap \text{Gen}(\mu_B)$, which is the set of $\Gamma(G)$ -generic points in B_G , has μ_B measure one. Hence, the set of non-generic points in B_G has μ_B measure zero. Since μ_B is a positive mixture of the μ_t 's for $t \in B$, it follows that the set of non-generic points in B_G has μ_t measure zero for every $t \in B$.

Consider the list of K_G words $s_0 \dots s_n$ with $s_n \in B$ and $s_i \in \text{tran}$ for all $i < n$. The cylinder sets $\langle s_0 \dots s_n \rangle$ associated with these words provides a countable decomposition of $\{\mathbf{s} : \text{End}(\mathbf{s}) = B\}$. If $\text{End}(\mathbf{s}) = B$ then $\mathbf{s} \in \langle s_0 \dots s_n \rangle$ with n the first entry time into B . To show that the non-generic points with $\text{End}(\mathbf{s}) = B$ have μ_s measure zero

it suffices to show that the intersection with each of these cylinder sets has measure zero. If $s_0 \neq s$ then $\mu_s \langle s_0 \dots s_n \rangle = 0$. So we may assume that $s_0 = s$. It suffices to show that the conditional probability that \mathbf{s} is non-generic given $\mathbf{s} \in \langle s_0 \dots s_n \rangle$ is zero. Observe that \mathbf{s} is $\Gamma(G)$ -generic iff $S(\mathbf{s})$ is $\Gamma(G)$ -generic. Hence, it suffices to show that zero is the conditional probability that $\mathbf{t} = S^n(\mathbf{s})$ is non-generic, given $\mathbf{s} \in \langle s_0 \dots s_n \rangle$, but because the process is Markov and $s_0 = s$, this is the μ_{s_n} measure that \mathbf{t} is non-generic. Since $s_n \in B$ and $\mathbf{t} \in B_G$, the μ_{s_n} probability that \mathbf{t} is non-generic is zero.

It follows that the set \mathbf{s} such that either $End(\mathbf{s})$ is not terminal or $End(\mathbf{s}) = B$ for some terminal basic set B but \mathbf{s} is not generic has μ_s measure zero.

□

The subshift associated with a relation G and a stochastic cover $\Gamma(G)$ provides the motivating example of a tractable system.

Corollary 3.9. *Let $\Gamma(G)$ be a stochastic cover of a relation G on a finite set K and let v be a positive distribution on K . With μ_v as background measure, the system (K_G, S) is tractable with μ_B the ergodic measure on B_G for each terminal G basic set B .*

Proof: TRAC 1 follows from Proposition 3.1 (b). TRAC 2 follows from Theorem 3.8. For each terminal basic set B the basin $G(B_G) = \{\mathbf{s} : End(\mathbf{s}) = B\}$ contains $Gen(\mu_B)$ by Lemma 3.7. This implies TRAC 3. Since distinct basic sets are disjoint, TRAC 4 holds.

□

Our applications will use the so-called *two alphabet model* as described in Proposition 1.4 of [2]. Assume that we are given two finite sets K^* and K together with a map $\gamma : K^* \rightarrow K$ and a relation $J : K^* \rightarrow K$. We assume that J is a *surjective relation* meaning that $p_1(J) = K^*$ and $p_2(J) = K$, or, equivalently, $Dom(J) = K^*$ and $Dom(J^{-1}) = K$. We define the relations $G = \gamma \circ J^{-1}$ on K and $G^* = J^{-1} \circ \gamma$ on K^* . Observe that $Dom(G) = K$ and $Dom(G^*) = K^*$. Since $\gamma \circ G^* = G \circ \gamma$, γ maps G^* to G .

Proposition 3.10. *For finite sets K, K^* assume that $\gamma : K^* \rightarrow K$ is a map and $J : K^* \rightarrow K$ is a surjective relation. Let $G = \gamma \circ J^{-1}$ and $G^* = J^{-1} \circ \gamma$.*

The map γ induces a bijection from the set of G^ basic sets to the set of G basic sets. If B^* is a G^* basic set, then $B = \gamma(B^*)$ is the associated G basic set. If B is a G basic set then $B^* = \gamma^{-1}(B) \cap J^{-1}(B)$ is the G^* basic set to which it is associated.*

If B^* and B are associated basic sets then the following are equivalent:

- (i) B^* is terminal, i.e. $\mathcal{O}G^*(B^*) \subset B^*$.
- (ii) $G^*(B^*) = B^*$.
- (iii) B is terminal, i.e. $\mathcal{O}G(B) \subset B$.
- (iv) $G(B) = B$.
- (v) $J^{-1}(B) = B^*$.

Proof: Observe first that

$$(3.6) \quad \mathcal{O}G = \gamma \circ (\mathcal{O}G^* \cup 1_{K^*}) \circ J^{-1}, \quad \text{and} \quad \mathcal{O}G^* = J^{-1} \circ (\mathcal{O}G \cup 1_K) \circ \gamma.$$

Since γ maps G^* to G , $\gamma(|\mathcal{O}G^*|) \subset |\mathcal{O}G|$, and for each G^* basic set B^* there is a unique G basic set B such that $\gamma(B^*) \subset B$.

Let $s \in |\mathcal{O}G|$, i.e. $(s, s) \in \mathcal{O}G$. By (3.6) there exists $s^*, t^* \in K^*$ such that $\gamma(s^*) = s$, $s \in J(t^*)$ and $(t^*, s^*) \in \mathcal{O}G^* \cup 1_{K^*}$. Hence, $(s^*, s^*) \in (\mathcal{O}G^* \cup 1_{K^*}) \circ J^{-1} \circ \gamma = \mathcal{O}G^*$. Thus, $s^* \in |\mathcal{O}G^*|$ and so $\gamma(|\mathcal{O}G^*|) = |\mathcal{O}G|$. Thus,

$$(3.7) \quad |\mathcal{O}G| = \bigcup \{\gamma(B^*) : B^* \text{ a } G^* \text{ basic set}\}.$$

Now let $s^* \in |\mathcal{O}G^*|$ with B^* the G^* basic set containing it. By (3.6) again, there exist $s, t \in K$ with $\gamma(s^*) = s$, $t \in J(s^*)$ and $(s, t) \in (\mathcal{O}G \cup 1_K)$. and $(t, s) \in \gamma \circ J^{-1} = G$. Hence, s and t are in $|\mathcal{O}G|$ and lie in the same basic set B . It follows that $s^* \in \gamma^{-1}(B) \cap J^{-1}(B)$. Thus,

$$(3.8) \quad |\mathcal{O}G^*| \subset \bigcup \{\gamma^{-1}(B) \cap J^{-1}(B) : B \text{ a } G \text{ basic set}\}.$$

Finally, suppose that $s^*, t^* \in \gamma^{-1}(B) \cap J^{-1}(B)$ for some G basic set B . Then $s = \gamma(s^*) \in B$ and there exists $t \in B$ such that $t \in J(t^*)$. Since $s, t \in B$, $(s, t) \in \mathcal{O}G$. So $(s^*, t^*) \in J^{-1} \circ (\mathcal{O}G) \circ \gamma \subset \mathcal{O}G^*$. Applying this first to the case $s^* = t^*$ we see that $s^*, t^* \in |\mathcal{O}G^*|$. This yields equality in (3.8). Furthermore, applying this to the pair (s^*, t^*) and to the pair with the order reversed we see that s^* and t^* lie in the same G^* basic set. It follows that distinct basic sets of G^* are mapped by γ onto distinct basic sets of G .

Now assume that B^* and B are associated basic sets. Since $B \subset |\mathcal{O}G|$ it follows that for $s \in B$, there is a sequence s_0, \dots, s_n with $n \geq 1$ such that $s_0 = s_n = s$ and $s_i \in G(s_{i-1})$ for $i = 1, \dots, n$. The members of the sequence all lie in the same $\mathcal{O}G \cap \mathcal{O}G^{-1}$ equivalence class, namely B . In particular, $B \subset G(B)$. Similarly, $B^* \subset G(B^*)$. Hence, (i) \Leftrightarrow (ii) and (iii) \Leftrightarrow (iv).

$G^*(B^*) = J^{-1}(\gamma(B^*)) = J^{-1}(B)$ since $B = \gamma(B^*)$. So (ii) \Leftrightarrow (v).

Because $\gamma(J^{-1}(B)) = G(B)$ and $\gamma(B^*) = B$, it follows that (v) \Rightarrow (iv). Conversely, $\gamma(J^{-1}(B)) = B$ implies that $J^{-1}(B) \subset \gamma^{-1}(B) \cap J^{-1}(B) = B^* \subset J^{-1}(B)$ and so $J^{-1}(B) = B^*$. Thus, (iv) \Rightarrow (v).

□

We will use a *special two-alphabet model* where J as well as γ is a map. In our applications the finite sets K^* and K will be identified with certain subsets of an ambient space. Each $s^* \in K^*$ will be contained in a unique $s = J(s^*) \in K$ with s the union of the s^* 's contained therein. Thus, J will be a surjective map.

To define a stochastic cover, we use a map $\nu : K^* \rightarrow [0, 1]$ such that

$$(3.9) \quad \begin{aligned} \nu(s^*) &> 0, \quad \text{for all } s^* \in K^*. \\ \sum \{\nu(s^*) : J(s^*) = s\} &= 1, \quad \text{for all } s \in K. \end{aligned}$$

Thus, ν is a positive distribution on the elements of $J^{-1}(s)$ for each $s \in K$. We will refer to such a map ν as *distribution data* for the special two alphabet model.

For the relations $G = \gamma \circ J^{-1}$ and $G^* = J^{-1} \circ \gamma$ we define from the distribution data ν the stochastic covers

$$(3.10) \quad \begin{aligned} \Gamma_{s_2 s_1} &= \sum \{\nu(s^*) : J(s^*) = s_1, \text{ and } \gamma(s^*) = s_2\}, \\ \Gamma_{s_2^* s_1^*}^* &= \begin{cases} \nu(s_2^*) & \text{if } J(s_2^*) = \gamma(s_1^*), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, for Γ , given $s_1 \in K$ we choose $s^* \in J^{-1}(s_1)$ with probability $\nu(s^*)$ and then apply γ . For Γ^* , given s_1^* we apply γ to get $s = \gamma(s_1^*)$ and then choose $s_2^* \in J^{-1}(s)$ with probability $\nu(s_2^*)$.

We have $\Gamma_{s_2 s_1} > 0$ iff $(s_1, s_2) \in G$ since otherwise we are summing over the empty set with sum equal to zero by convention. Clearly, $\Gamma_{s_2^* s_1^*}^* > 0$ iff $(s_1^*, s_2^*) \in G^*$.

$$(3.11) \quad \begin{aligned} \sum_{s_2 \in K} \Gamma_{s_2 s_1} &= \sum \{\nu(s^*) : J(s^*) = s_1\} = 1. \\ \sum_{s_2^* \in K^*} \Gamma_{s_2^* s_1^*}^* &= \sum \{\nu(s_2^*) : J(s_2^*) = \gamma(s_1^*)\} = 1. \end{aligned}$$

We will call Γ and Γ^* the *stochastic covers induced by ν* .

Proposition 3.11. *Let $\gamma : K^* \rightarrow K$ and $J : K^* \rightarrow K$ be a special two alphabet model with distribution data ν . Let Γ and Γ^* be the stochastic covers of $G = \gamma \circ J^{-1}$ and $G^* = J^{-1} \circ \gamma$ induced by ν .*

If $v : K \rightarrow [0, 1]$ is a stationary distribution for Γ then $v^ : K^* \rightarrow [0, 1]$ defined by $v^*(s^*) = v(J(s^*))\nu(s^*)$ is a stationary distribution for Γ^* .*

Proof: Since v is stationary for Γ , $\sum_{s_1} v(s_1)\Gamma_{s_2 s_1} = v(s_2)$.

Fix $s_2^* \in K^*$ and let $s_2 = J(s_2^*)$

$$\begin{aligned}
 (\Gamma \cdot v^*)(s_2^*) &= \Sigma_{s_1^*} \{ \nu(s_1^*) v(J(s_1^*)) \nu(s_2^*) : J(s_2^*) = \gamma(s_1^*) \} \\
 (3.12) \quad &= \nu(s_2^*) \Sigma_{s_1} v(s_1) \Sigma \{ \nu(s_1^*) : s_2 = \gamma(s_1^*) \text{ and } J(s_1^*) = s_1 \} \\
 &= \nu(s_2^*) \Sigma_{s_1} v(s_1) \Gamma_{s_2 s_1} = \nu(s_2^*) v(s_2) = v^*(s_2^*),
 \end{aligned}$$

as required.

□

Remarks: (a) Alternatively, for v^* , $v^*(s) = \Sigma \{ v^*(s^*) : J(s^*) = s \}$ is equal to $v(s)$, and given $J(s^*) = s$ the conditional probability of s^* is $\nu(s^*)$.

(b) If B and B^* are associated terminal basic sets, then the positive stationary distribution for Γ with support B given by Proposition 3.4 (b) satisfies for $s_2 \in K$:

$$(3.13) \quad \Sigma_{s^* \in B^* \cap \gamma^{-1}(s_2)} v_B(J(s^*)) \cdot \nu(s^*) = v_B(s_2).$$

Note that the sum is empty and so equals 0 when $s_2 \notin B$.

From (3.5) and (3.10) we have, for $s^* \in K^*$:

$$(3.14) \quad \mu_{s^*} \langle s_0^* \dots s_n^* \rangle = \nu(s_1^*) \cdot \dots \cdot \nu(s_n^*)$$

if $s_0^* \dots s_n^*$ is a $K_{G^*}^*$ word with $s_0^* = s^*$, and = 0 otherwise. For $s \in K$ we define the measure on K^*

$$(3.15) \quad \mu_s = \Sigma \{ \nu(s^*) \mu_{s^*} : J(s^*) = s \}.$$

Thus,

$$(3.16) \quad \mu_s \langle s_0^* \dots s_n^* \rangle = \nu(s_0^*) \cdot \nu(s_1^*) \cdot \dots \cdot \nu(s_n^*)$$

if $s_0^* \dots s_n^*$ is a $K_{G^*}^*$ word with $J(s_0^*) = s$, and = 0 otherwise.

If $B \subset K$ and $B^* \subset K^*$ are associated terminal basic sets, let v_B be the positive stationary distribution for Γ with support B given by Proposition 3.4 (b). From Proposition 3.5 (b) and Proposition 3.11 it follows that the associated ergodic measure on B^* is given by

$$(3.17) \quad \mu_{B^*} = \Sigma_{s^* \in B^*} v_B(J(s^*)) \nu(s^*) \mu_{s^*} = \Sigma_{s \in B} v_B(s) \mu_s.$$

Thus,

$$(3.18) \quad \mu_{B^*} \langle s_0^* \dots s_n^* \rangle = v_B(J(s_0^*)) \cdot \nu(s_0^*) \cdot \nu(s_1^*) \cdot \dots \cdot \nu(s_n^*)$$

if $s_0^* \dots s_n^*$ is a $K_{G^*}^*$ word with $s_0^* \in B^*$, and = 0 otherwise.

4. SIMPLICIAL DYNAMICAL SYSTEMS

An n simplex s in a vector space is the convex hull of a finite, affinely independent set $V(s) = \{v_0, \dots, v_n\}$, the *vertices* of s . A simplex s_1 with vertices a (proper) nonempty subset of $V(s)$ is a (proper) *face* of s and we write $s_1 \leq s$ (or $s_1 < s$ for a proper face). In particular, each vertex is a 0 simplex face of s . The union of the proper faces is the *boundary* ∂s . We denote by s° the open simplex $s \setminus \partial s$. It is the interior of s in the n dimensional affine subspace generated by the vertices.

A *simplicial complex* K is a nonempty finite collection of closed simplices such that $s_1 < s$ and $s \in K$ implies $s_1 \in K$ and $s_1, s_2 \in K$ with $s_1 \cap s_2$ nonempty implies $s_1 \cap s_2$ is a common face of s_1 and s_2 . We let $V(K)$ denote the set $\bigcup_{s \in K} V(s)$ of vertices of K . The polyhedron associated with K , denoted $X(K)$, is the union $\bigcup_{s \in K} s$. A space X is a polyhedron when there exists a simplicial complex K such that when $X(K) = X$, in which case, we say that K is a triangulation of a polyhedron X . Every point $x \in X$ can then be written uniquely as

$$(4.1) \quad x = \sum_{v \in V(K)} b_v(x) v$$

with $b_v(x) \geq 0$ for all v , $\sum_{v \in V(K)} b_v(x) = 1$, and $\{v : b_v(x) > 0\}$ the vertices of a simplex of K called the *carrier* of x . The numbers $\{b_v(x) : v \in V(K)\}$ are called the *barycentric* coordinates of x .

We define the metric d_K on X as the L^1 distance between barycentric coordinates:

$$(4.2) \quad d_K(x, y) = \sum_{v \in V(K)} |b_v(x) - b_v(y)|.$$

So the distance between distinct vertices of K is 2 and this is the d_K diameter of X if K is non-trivial.

In general, if we have a fixed background metric d on a polyhedron X then the *mesh* of a triangulation K of X , denoted $\text{mesh}_d K$ is the maximum of the d -diameters of the simplices of K . Of course, if we use d_K then the mesh is 2.

A nonempty subset K_1 of K is a complex when $s_1 < s$ and $s \in K_1$ implies $s_1 \in K_1$ in which case we call K_1 a *subcomplex* of K . If $X_1 \subset X(K)$ we will say that X_1 is triangulated by K when $X_1 = X(K_1)$ for K_1 a subcomplex of K .

We will identify a simplex s of K with the subcomplex consisting of all of the faces of s .

If K^* and K are triangulations of X , we call K^* a *subdivision* of K if every simplex of K is a union of simplices of K^* and so is a subcomplex

of K^* . In that case, we let $J : K^* \rightarrow K$ denote the inclusion relation:

$$(4.3) \quad J = \{(s^*, s) : s^* \subset s\}.$$

Notice that J is a surjective relation, i.e. $\text{Dom}(J) = K^*$ and $\text{Dom}(J^{-1}) = K$. If no simplex of K^* meets two disjoint simplices of K then we call K^* a *proper subdivision* of K . Clearly, if K^* is a proper subdivision of K then any subdivision of K^* is a proper subdivision of K .

A map $\gamma : K_1 \rightarrow K_2$ between simplicial complexes is a *simplicial map* if $\gamma(V(s)) = V(\gamma(s))$ for all $s \in K_1$. Clearly, the dimension of $\gamma(s)$ is at most that of s with $\dim \gamma(s) = \dim s$ iff γ is injective on the set of vertices $V(s)$. When this holds for all $s \in K_1$ we call γ *non-degenerate*.

The associated p.l. (= piecewise linear) map $g : X(K_1) \rightarrow X(K_2)$ is obtained by extending linearly on each simplex the map on the vertices. That is,

$$(4.4) \quad g(x) = \sum_{v \in V(K_1)} b_v(x) \gamma(v).$$

In general, a map $g : X_1 \rightarrow X_2$ between polyhedra is called a p.l. map when it is associated with a simplicial map between triangulations of X_1 and X_2 . While it is obvious that the composition of simplicial maps is simplicial, it is not so obvious that the composition of p.l. maps is p.l. It is, nonetheless true, see, e.g. [5] Theorem 1.11. It is also shown there that a map $g : X_1 \rightarrow X_2$ between polyhedra is p.l. exactly when $g \subset X_1 \times X_2$ is a polyhedron.

Definition 4.1. A simplicial dynamical system is a simplicial map $\gamma : K^* \rightarrow K$ with K^* a proper subdivision of K .

From such a simplicial dynamical system $\gamma : K^* \rightarrow K$ together with the inclusion relation $J : K^* \rightarrow K$ we obtain a two alphabet model with relations $G = \gamma \circ J^{-1}$ on K and $G^* = J^{-1} \circ \gamma$ on K^* .

The approximation results will require a bit more background. A *derived subdivision* (or first derived subdivision) K' of a complex K is obtained by choosing a vertex $v(s) \in s^\circ$ for each $s \in K$. The set of vertices $\{v(s_0), \dots, v(s_n)\}$ spans a simplex of K' iff they can be renumbered so that $s_0 < s_1 < \dots < s_n$ in K . The derived subdivision is a proper subdivision of K .

If L is a subcomplex of K then L' is a subcomplex of K' . Hence, if Y is triangulated by K then it is triangulated by K' .

For Y a subset triangulated by K we let

$$(4.5) \quad \begin{aligned} N^\circ(Y, K) &= \bigcup \{s^\circ : s \in K \text{ and } s \cap Y \neq \emptyset\} \\ &= \{x : b_v(x) > 0 \text{ for some vertex } v \text{ of } K \text{ contained in } Y\}. \end{aligned}$$

Thus, $N^\circ(Y, K)$ is an open subset of $X(K)$ which contains Y .

Lemma 4.2. *If Y_1 and Y_2 are triangulated by K and K' is a derived subdivision then*

$$(4.6) \quad N^\circ(Y_1, K') \cap N^\circ(Y_2, K') = N^\circ(Y_1 \cap Y_2, K').$$

For example, if $s_1, s_2 \in K$ then $N^\circ(s_1, K') \cap N^\circ(s_2, K') = N^\circ(s_1 \cap s_2, K')$. If s_1 and s_2 are disjoint then $N^\circ(s_1, K')$ and $N^\circ(s_2, K')$ are disjoint.

Proof: Let the carrier of x in K' be the simplex spanned by $v(s_0), \dots, v(s_k)$ with $s_{i-1} < s_i$ for $i = 1, \dots, k$.

Assume $x \in N^\circ(Y_1, K')$. Since Y_1 is triangulated by K' , $v(s_{i_1}) \in Y_1$ for some i_1 between 0 and k . Since Y_1 is triangulated by K , $s_{i_1} \subset Y_1$. If $x \in N^\circ(Y_2, K')$ as well, then, similarly, $s_{i_2} \subset Y_2$ for some i_2 between 0 and k . Without loss of generality we may assume $i_1 \leq i_2$ so that $s_{i_1} \subset Y_1 \cap Y_2$ and so $x \in N^\circ(Y_1 \cap Y_2, K')$.

□

The following is a version of the standard p.l. approximation theorem.

Theorem 4.3. *Let X and Y be polyhedra equipped with metrics d_X and d_Y , and let K be a triangulation of X . If $f : Y \rightarrow X$ is a continuous map then there exists $\delta > 0$ so that if L is any triangulation of Y with $\text{mesh}_{d_Y} L < \delta$ then, for L' any derived subdivision of L , there exists a simplicial map $\gamma : L' \rightarrow K$ with $d(f(x), g(x)) \leq 2\text{mesh}_{d_X} K$ for all $x \in Y$, where g is the p.l. map associated with γ .*

Proof: $\{N^\circ(s, K') : s \in K\}$ is an open cover of X . Let $\ell > 0$ be a Lebesgue number for the cover and $\delta > 0$ be an ℓ modulus of uniform continuity for f . Hence, if $\text{mesh}_{d_Y} L < \delta$ then for any $t \in L$ there exists $s \in K$ such that $f(t) \subset N^\circ(s, K')$. Let $\sigma_f(t)$ be the intersection of all such simplices s . By Lemma 4.2 $f(t) \subset N^\circ(\sigma_f(t), K')$ and $\sigma_f(t) \in K$ is the smallest such simplex. This minimality implies that $t \mapsto \sigma_f(t)$ is incidence-preserving, i.e. $t_1 \leq t_2$ implies $\sigma_f(t_1) \leq \sigma_f(t_2)$.

Define γ by associating to the vertex $v(t) \in L'$ an arbitrary vertex in $\sigma_f(t)$. If $t_0 < t_1 \cdots < t_k$ then $\sigma_f(t_0) \leq \sigma_f(t_1) \cdots \leq \sigma_f(t_k)$ and so $\gamma(v(t_0)), \dots, \gamma(v(t_k))$ are all vertices of $\sigma_f(t_k)$. Hence, each simplex of L' is mapped to a simplex of K . That is, γ is a simplicial map. If the carrier of $x \in Y$ is the simplex with vertices $v(t_0), \dots, v(t_k)$ then $g(x) \in \sigma_f(t_k)$ and $f(x) \in f(t_k) \subset N^\circ(\sigma_f(t_k), K')$. Hence, $d_X(g(x), f(x)) \leq 2\text{mesh}_{d_X} K$.

□

When $Y = X$ we can choose L to be a subdivision of K' which will ensure that it is a proper subdivision of K . So we obtain the following.

Corollary 4.4. *Let X be a polyhedron equipped with a metric d_X and let K be any triangulation of X . If f is a continuous map on X , then there exists a simplicial dynamical system $\gamma : K^* \rightarrow K$ such that $d(f(x), g(x)) \leq 2\text{mesh}_{d_X} K$ for all $x \in X$, where g is the p.l. map associated with γ .*

□

The maps constructed above are called *p.l. roundoffs* for f .

A p.l. d -manifold is a polyhedron X such that every point of X has a neighborhood which is p.l. homeomorphic to a simplex of dimension d . The following result, Theorem 9.3 of [2], allows us to refine Corollary 4.4 to obtain approximation by non-degenerate simplicial dynamical systems in the manifold case. The proof is rather technical and so we will simply refer the reader to [2].

Theorem 4.5. *Let X be a p.l. manifold triangulated by a complex L . Assume that $\gamma : K^* \rightarrow K$ is a simplicial dynamical system with K a subdivision of L . There exists a subdivision K_1^* of K^* and a non-degenerate simplicial map $\gamma_1 : K_1^* \rightarrow K$ such that $d_L(g(x), g_1(x)) \leq 4\text{mesh}_{d_L} K$ for all $x \in X$, where g and g_1 are the p.l. maps associated with γ and γ_1 , respectively.*

□

Since any subdivision of a proper subdivision is a proper subdivision, $\gamma_1 : K_1^* \rightarrow K$ is a simplicial dynamical system.

While the dynamics of simplicial dynamical systems with degeneracies is considered in [2], we will restrict attention here to the special case of a non-degenerate system on a p.l. d -manifold with $d \geq 1$. Of course, a complex of dimension zero is just a finite set on which the dynamics is simple to describe directly.

The key result is that the local inverses of a non-degenerate simplicial dynamical system are uniformly contracting.

On the spaces \mathbb{R}^n we use the L^1 norm and let $\mathbb{R}_0^n = \{a \in \mathbb{R}^n : \sum_i a_i = 0\}$.

Lemma 4.6. *Let (P_{ij}) be an $(m+1) \times (d+1)$ non-negative matrix with $\sum_i P_{ij} = 1$ for all j . Let $P : \mathbb{R}_0^{d+1} \rightarrow \mathbb{R}_0^{m+1}$ be the associated linear map between the subspaces with coordinate sums equal to 0. If there exists $\theta > 0$ such that for every pair $j_1, j_2 \in \{0, \dots, d\}$ there exists*

$i_0 \in \{0, \dots, m\}$ such that $P_{i_0 j_1}, P_{i_0 j_2} \geq \theta$ then the norm $\|P\|$ of the linear map is bounded by $1 - (\theta/d)$.

Proof: Assume $a \in R_0^{d+1}$ and $a \neq 0$. Let a_+, a_- be the sum of the positive and of the negative components, respectively. Thus, $\|a\| = a_+ - a_-$ and $0 = a_+ + a_-$. That is, $a_+ = -a_- = \|a\|/2$ and there are at most d nonzero components of either type. Hence, there exist j_1, j_2 such that $a_{j_1}, -a_{j_2} \geq \|a\|/2d$. Choose i_0 as above for j_1, j_2 so that

$$\begin{aligned}
 & P_{i_0 j_1} |a_{j_1}| + P_{i_0 j_2} |a_{j_2}| - \theta \|a\|/d = \\
 & P_{i_0 j_1} a_{j_1} - P_{i_0 j_2} a_{j_2} - \theta \|a\|/d \geq \\
 (4.7) \quad & \max(P_{i_0 j_1} a_{j_1} + P_{i_0 j_2} a_{j_2}, -P_{i_0 j_1} a_{j_1} - P_{i_0 j_2} a_{j_2}) = \\
 & |P_{i_0 j_1} a_{j_1} + P_{i_0 j_2} a_{j_2}|.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (4.8) \quad & \|Pa\| = \sum_i |\sum_j P_{ij} a_j| \leq \\
 & \sum_{i,j} P_{ij} |a_j| - \theta \|a\|/d = (1 - (\theta/d)) \|a\|.
 \end{aligned}$$

□

If K^* is a subdivision of K then we define

$$(4.9) \quad \theta(K^*, K) = \min\{b_v(v^*) : v^* \in V(K^*), v \in V(K), \text{ and } b_v(v^*) > 0\}.$$

That is, $\theta(K^*, K)$ is the minimum positive K barycentric coordinate of a vertex of K^* .

Lemma 4.7. *Assume K^* is a proper subdivision of K and that $v_1^*, v_2^* \in V(K^*)$ span a simplex in K^* . There exists $v_0 \in V(K)$ such that $b_{v_0}(v_1^*), b_{v_0}(v_2^*) \geq \theta(K^*, K)$.*

Proof: If, instead, $\{v : b_v(v_1^*) > 0\}$ and $\{v : b_v(v_2^*) > 0\}$ are disjoint, then the carriers in K of v_1^* and v_2^* are disjoint and so the 1-simplex of K^* which they span meets disjoint simplices of K . This means that K^* is not a proper subdivision.

□

Proposition 4.8. *Let $\gamma : K^* \rightarrow K$ be a non-degenerate simplicial dynamical system on a polyhedron X of positive dimension d . For $s^* \in K^*$ let \bar{g}_{s^*} be the affine isomorphism from $\gamma(s^*) \rightarrow s^* \subset X$ which is the inverse of the restriction of the p.l. map g from s^* to $\gamma(s^*)$. With respect to the metric d_K the map \bar{g}_{s^*} is a contraction. For x_1, x_2 points in the K simplex $\gamma(s^*)$*

$$(4.10) \quad d_K(\bar{g}_{s^*}(x_1), \bar{g}_{s^*}(x_2)) \leq (1 - (\theta(K^*, K)/d)) d_K(x_1, x_2).$$

Proof: List the vertices of K beginning with those of $\gamma(s^*)$ so that $V(K) = \{v_0, \dots, v_m\}$ and $V(\gamma(s^*)) = \{v_0, \dots, v_n\}$, so that $n \leq d$. Define $P_{ij} = b_{v_i}(\bar{g}_{s^*}(v_j))$. For x_1, x_2 in $\gamma(s^*)$ let a_1, a_2 be the corresponding vectors of $\gamma(s^*)$ barycentric coordinates. Observe that the remaining K barycentric coordinates for x_1 and x_2 are all zero. Hence the barycentric coordinates of $\bar{g}_{s^*}(x_1), \bar{g}_{s^*}(x_2)$ are given by Pa_1, Pa_2 . Hence,

$$(4.11) \quad \begin{aligned} d_K(x_1, x_2) &= \|a_1 - a_2\|, \\ d_K(\bar{g}_{s^*}(x_1), \bar{g}_{s^*}(x_2)) &= \|Pa_1 - Pa_2\| = \|P(a_1 - a_2)\|. \end{aligned}$$

Since $a_1 - a_2 \in \mathbb{R}_0^{n+1}$ with $n \leq d$, (4.10) follows from Lemmas 4.7 and 4.6. Note that the vertices $\bar{g}_{s^*}(v_j)$, $j = 0, \dots, n$ are the K^* vertices of s^* and so any two of them span a simplex of K^* .

□

Theorem 4.9. *Let $\gamma : K^* \rightarrow K$ be a non-degenerate simplicial dynamical system on a polyhedron X of positive dimension d . Let g be the associated p.l. map on X . Let $J : K^* \rightarrow K$ be the inclusion relation and $G^* = J^{-1} \circ \gamma$ the associated relation on the finite set K^* .*

$$(4.12) \quad H_\gamma = \{(\mathbf{s}^*, x) \in K_{G^*}^* \times X : g^i(x) \in s_i^* \text{ for } i \in \mathbb{Z}_+\}$$

is a continuous map from $K_{G^}^*$ onto X which maps the shift S to g .*

Proof: It is easy to check that H_γ is a closed subset of $K_{G^*}^* \times X$ which is $+$ invariant with respect to the map $S \times g$.

If $x \in X$, we choose $s_0^* \in K^*$ so that $x \in s_0^*$. Inductively, suppose we have s_0^*, \dots, s_n^* such that $(s_{i-1}^*, s_i^*) \in G^*$ and $g^i(x) \in s_i^*$ for $i = 1, \dots, n$. Since $g^{n+1}(x) \in \gamma(s_n^*)$ we can choose $s_{n+1}^* \in \gamma(s_n^*)$ with $g^{n+1}(x) \in s_{n+1}^*$ and so $(s_n^*, s_{n+1}^*) \in J^{-1} \circ \gamma = G^*$. We thus obtain, by induction, a sequence $\mathbf{s} \in K_{G^*}^*$ such that $(\mathbf{s}, x) \in H_\gamma$.

To complete the proof it suffices to show that H_γ is a map. That is, for $\mathbf{s} \in K_{G^*}^*$ there is a unique x such that $(\mathbf{s}, x) \in H_\gamma$.

Consider the sequence of local inverses $\bar{g}_{s_i^*} : \gamma(s_i^*) \rightarrow s_i^*$ for $i = 0, \dots, n$. Because $(s_i^*, s_{i+1}^*) \in G^*$ it follows that the image, s_{i+1}^* of $\bar{g}_{s_{i+1}^*}$ is contained in the domain $\gamma(s_i^*)$ of $\bar{g}_{s_i^*}$. So the sequence of images

$$(4.13) \quad \{\bar{g}_{s_0^*} \circ \bar{g}_{s_1^*} \circ \dots \circ \bar{g}_{s_{n-1}^*} \circ \bar{g}_{s_n^*}(g(s_n^*))\}$$

is a decreasing sequence of subsets of subsets of s_0^* . By Proposition 4.8 the diameter d_K of the set indexed by n is at most $2 \times (1 - (\theta(K^*, K)/d))^{n+1}$. Hence, the intersection is a single point x . Clearly, x is the unique point such that $(\mathbf{s}, x) \in H_\gamma$.

□

We can use γ to define a sequence of refinements of K . For each $s^* \in K^*$ we can use the affine isomorphism $\gamma : s^* \rightarrow \gamma(s^*) \in K$ to pull back the K^* triangulation of $\gamma(s^*)$ to obtain a triangulation of s^* . Doing this for every s^* in K , we obtain a proper refinement K^{**} of K^* with a simplicial map $\gamma^* : K^{**} \rightarrow K^*$ such that γ and γ^* have the same p.l. map g on X . We repeat the procedure, inductively defining a simplicial dynamical system $\gamma^{*n} : K^{*(n+1)} \rightarrow K^{*n}$ with the p.l. map g for all n . If s_0^*, \dots, s_n^* is a $K_{G^*}^*$ word then $\bar{g}_{s_0^*} \circ \dots \circ \bar{g}_{s_{n-1}^*}(s_n^*)$ is a simplex of $K^{*(n+1)}$ and every simplex of $K^{*(n+1)}$ is obtained this way. It follows from Proposition 4.8 that

$$(4.14) \quad mesh_{d_K} K^{*n} \leq 2(1 - (\theta(K^*, K)/d))^n.$$

Let $Z_0 \subset X$ denote the polyhedron triangulated by the $d - 1$ skeleton of K^* , i.e. the union of all of the simplices of dimension less than d . A point x lies in $X \setminus Z_0 = \bigcup \{(s^*)^\circ\}$, with s^* varying over the d -dimensional simplices of K^* , exactly when its K^* carrier is a d -dimensional simplex and so the carrier is the unique simplex of K^* which contains x . Since K^* is a subdivision of K , we see that Z_0 contains all of the simplices of K with dimension less than d . Inductively, if $Z_n \subset X$ is the polyhedron triangulated by the $d - 1$ skeleton of $K^{*(n+1)}$ then $Z_{n-1} \subset Z_n$. We thus obtain an increasing sequence of dimension $d - 1$ polyhedra in X . A point $x \in X \setminus Z_n$ iff the $K^{*(n+1)}$ carrier of x is d -dimensional, or, equivalently, for $i = 0, \dots, n$, the K^* carrier of $g^i(x)$ is d -dimensional. Thus, for $i = 0, \dots, n$ each $g^i(x)$ is contained in a unique simplex of K^* .

Because of Theorem 4.5, we are primarily interested here in dynamical systems on a p.l. manifold. It will be convenient to consider a slightly more general situation. Call a polyhedron X *everywhere d -dimensional* when it is a union of d -dimensional closed simplices. If K triangulates X then every simplex of K is the face of a d -dimensional simplex.

For the rest of the section we will assume that $\gamma : K^* \rightarrow K$ is a non-degenerate simplicial dynamical system on an everywhere d -dimensional polyhedron with d positive. We let ${}^dK^*$ and dK denote the set of d -dimensional simplices of K^* and K respectively. Every d -simplex of K^* is contained in a unique d -simplex of K . That is, the inclusion relation restricts to a map J from ${}^dK^*$ onto dK . Because γ is non-degenerate, it restricts to a map from ${}^dK^*$ to dK . The relations $G^* = J^{-1} \circ \gamma$ and $G = \gamma \circ J^{-1}$ restrict to relations on ${}^dK^*$ and dK . Notice that ${}^dK_{G^*}^*$ is a closed invariant subset of $K_{G^*}^*$ and $({}^dK_{G^*}^*, S)$ is a subshift of finite type.

Thus, $J : {}^dK^* \rightarrow {}^dK$ and $\gamma : {}^dK^* \rightarrow {}^dK$ is a special two-alphabet model with associated relations G^* and G .

Theorem 4.10. *Let $\gamma : K^* \rightarrow K$ be a non-degenerate simplicial dynamical system on an everywhere d -dimensional polyhedron X with g be the associated p.l. map on X . Let $J : {}^dK^* \rightarrow {}^dK$ be the inclusion map and $G^* = J^{-1} \circ \gamma$ the associated relation on the finite set ${}^dK^*$.*

$$(4.15) \quad H_\gamma = \{(\mathbf{s}^*, x) \in {}^dK_{G^*}^* \times X : g^i(x) \in s_i^* \text{ for } i \in \mathbb{Z}_+\}$$

is an almost one-to-one continuous map from ${}^dK_{G^*}^*$ onto X which maps the shift S to g .

Proof: The map labeled H_γ here is just the restriction of the map of Theorem 4.9 to the invariant subset ${}^dK_{G^*}^*$. For $x \in X$ we can repeat the inductive construction of the proof of Theorem 4.9, choosing a d dimensional $s_{i+1}^* \subset \gamma(s_i^*)$ with $g^{i+1}(x) \in s_{i+1}^*$. The resulting sequence \mathbf{s} lies in ${}^dK_{G^*}^*$. So the restriction is still onto.

Since X is everywhere d -dimensional, each $d-1$ dimensional polyhedron Z_n is nowhere dense in X and so $Z_\infty = \bigcup_n Z_n$ is of first category in X . For x in the dense set $X \setminus Z_\infty$, $H_\gamma^{-1}(x)$ is a singleton.

Now let $\mathbf{s}^* \in {}^dK_{G^*}^*$ and let n be a positive integer. The map $\bar{g}_{s_0^*} \circ \bar{g}_{s_1^*} \circ \dots \circ \bar{g}_{s_{n-1}^*}$ restricts to an affine isomorphism of s_n^* onto a d -dimensional simplex t of $K^{*(n+1)}$ contained in s_0^* . Each $t \cap Z_m$ for $m \geq n+1$ is at most $(d-1)$ -dimensional and so is nowhere dense in t . For $x \in t \setminus Z_\infty$, $H_\gamma^{-1}(x)$ is a singleton $\{\hat{\mathbf{s}}^*\}$ and the first n coordinates of $\hat{\mathbf{s}}^*$ agree with those of \mathbf{s}^* . It follows that $\text{Inj}_{H_\gamma} = \{\mathbf{s}^* : H_\gamma^{-1}(H_\gamma(\mathbf{s}^*)) \text{ is a singleton}\}$ is dense in ${}^dK_{G^*}^*$. Thus, H_γ is almost one-to-one on ${}^dK_{G^*}^*$.

□

Now we turn to measures. On each d -dimensional simplex s or s^* there is a normalized Lebesgue measure λ_s or λ_{s^*} . Since an affine mapping multiplies Lebesgue measure by a constant, the determinant, it follows that if $s^* \in {}^dK^*$ and $s = \gamma(s^*) \in {}^dK$, then $g_*(\lambda_{s^*}) = \lambda_s$, with g the linear map on s^* associated with γ . On the other hand, if $J(s^*) = s$, i.e. $s^* \subset s$ then $\lambda_s(s^*) > 0$ and the measure λ_{s^*} is just the restriction of λ_s to s^* normalized via division by $\lambda_s(s^*)$.

For the special two-alphabet model given by $\gamma : {}^dK^* \rightarrow {}^dK$ and $J : {}^dK^* \rightarrow {}^dK$, we define the distribution data $\nu : {}^dK^* \rightarrow [0, 1]$ by

$$(4.16) \quad \nu(s^*) = \lambda_s(s^*) \quad \text{with} \quad s = J(s^*).$$

From (3.16) we obtain for each $s \in {}^dK$ the measure μ_s on ${}^dK_{G^*}^*$ with

$$(4.17) \quad \mu_s \langle s_0^* \dots s_n^* \rangle = \nu(s_0^*) \cdot \nu(s_1^*) \cdot \dots \cdot \nu(s_n^*)$$

if $s_0^* \dots s_n^*$ is a ${}^dK_G^*$ word with $J(s_0^*) = s$, and $= 0$ otherwise. If we let

$$(4.18) \quad \langle s \rangle = \bigcup \{ \langle s^* \rangle \subset {}^dK_G^* : s^* \in J^{-1}(s) \},$$

then $|\mu_s| = \langle s \rangle$.

With v_0 a positive distribution on dK , let $\mu_0 = \sum_{s \in {}^dK} v_0(s) \mu_s$

Theorem 4.11. *The surjection H_γ of Theorem 4.10 is a μ_0 almost one-to-one map. If $s \in {}^dK$, then $H_\gamma(\langle s \rangle) = s$ and $(H_\gamma)_* \mu_s = \lambda_s$.*

Proof: It is clear that H_γ maps $\langle s \rangle$ into s . If $x \in s$ then there exists $s_0^* \in {}^dK^*$ with $x \in s_0^*$ and $s_0^* \subset s$. Beginning the inductive proof from Theorem 4.10 with s_0^* we obtain $\mathbf{s}^* \in \langle s \rangle$ such that $H_\gamma(\mathbf{s}^*) = x$.

For $s_0^* \dots s_n^*$ a ${}^dK_G^*$ word, let $s_k = J(s_k^*)$ for $k = 0, \dots, n$ so that $s_k = \gamma(s_{k-1}^*)$ for $k = 1, \dots, n$, and let $s_{n+1} = \gamma(s_n^*)$. We assume that $s_0 = s$ so that $\langle s_0^* \dots s_n^* \rangle \subset \langle s \rangle$.

Define

$$(4.19) \quad \begin{aligned} t_0 &= s_0^* = \bar{g}_{s_0^*}(s_1), \\ t_k &= \bar{g}_{s_0^*} \circ \dots \circ \bar{g}_{s_{k-1}^*}(s_k^*) = \bar{g}_{s_0^*} \circ \dots \circ \bar{g}_{s_k^*}(s_{k+1}), \end{aligned}$$

for $k = 1, \dots, n$.

We prove by induction that $\lambda_s(t_k) = \mu_s(\langle s_0^* \dots s_k^* \rangle)$.

For $k = 0$ this is the definition of $\nu(s_0^*)$.

For each $k \geq 1$, $\bar{g}_{s_0^*} \circ \dots \circ \bar{g}_{s_{k-1}^*}$ maps λ_{s_k} to $\lambda_{t_{k-1}}$ and by induction hypothesis $\lambda_s(t_{k-1}) = \nu(s_0^*) \dots \nu(s_{k-1}^*)$. The measure $\lambda_{t_{k-1}}$ is the restriction of λ_s divided by $\lambda_s(t_{k-1})$. By definition, $\lambda_{s_k}(s_k^*) = \nu(s_k^*)$. It follows that

$$\lambda_s(t_k) = \lambda_s(t_{k-1}) \cdot \nu(s_k^*) = \nu(s_0^*) \cdot \dots \cdot \nu(s_k^*) = \mu_s(\langle s_0^* \dots s_k^* \rangle).$$

As is shown in the proof of Theorem 4.10 H_γ is injective on the points of ${}^dK_G^* \setminus (H_\gamma)^{-1}(Z_\infty)$, and the map is open at these points. So H_γ restricts to a homeomorphism from ${}^dK_G^* \setminus (H_\gamma)^{-1}(Z_\infty)$ onto $X \setminus Z_\infty$. $\lambda_s(s \cap Z_\infty) = 0$ because Z_∞ is a countable union of lower dimensional polyhedra. Since the cylinders $\langle s_0^* \dots s_n^* \rangle$ comprise a basis for ${}^dK_G^*$, it will suffice to prove that $\mu_s((H_\gamma)^{-1}(Z_\infty)) = 0$.

In turn, it suffices to show that for each d -simplex $t \in K^{*n}$ the boundary ∂t lifts via H_γ to a set of measure zero. If $x \in \partial t$ and $\mathbf{s}^* \in \langle s \rangle$ with $x = H_\gamma(\mathbf{s}^*)$, then for $k > n$, t_k as defined above is a simplex of K^{*k} which contains x and so meets the boundary ∂t .

Since $\lambda_s(\partial t) = 0$ there exists for every $\epsilon > 0$ a $\delta > 0$ so that the λ_s measure of the δ neighborhood of ∂t is less than ϵ . For sufficiently large k , the mesh of K^{*k} is less than δ . So every \mathbf{s}^* with $H_\gamma(\mathbf{s}^*) \in \partial t$ has initial word $s_0^* \dots s_k^*$ whose cylinder is mapped by H_γ onto such a simplex t_k . The sum of the measures of all such t_k 's is less than ϵ and

it is equal to the μ_s measure of the sum of the cylinders. It follows that $\mu_s((H_\gamma)^{-1}(\partial t)) = 0$.

□

Now let B^* and B be associated terminal basic sets for $({}^dK, G)$ and $({}^dK^*, G^*)$. Let \bar{B}^* and \bar{B} consist of the subcomplexes of K^* and K consisting of all of the faces of the simplices of B^* and B , respectively. Because these are terminal sets, Proposition 3.10 implies that \bar{B}^* is a subdivision of \bar{B} . They are everywhere d -dimensional with $B^* = {}^d(\bar{B}^*)$ and $B = {}^d(\bar{B})$. Thus,

$$(4.20) \quad X(\bar{B}) = \bigcup \{s \in B\} = X(\bar{B}^*) = \bigcup \{s^* \in B^*\}.$$

There is a unique positive distribution v_B on B , i.e. for $s \in {}^dK$

$$(4.21) \quad \sum_{s^* \in B^* \cap \gamma^{-1}(s)} v_B(J(s^*)) \cdot \nu(s^*) = v_B(s).$$

Since the restriction $\gamma : \bar{B}^* \rightarrow \bar{B}$ is a non-degenerate simplicial dynamical system, Theorem 4.10 implies that $H_\gamma(B_{G^*}^*) = X(\bar{B}^*) = X(\bar{B})$.

Theorem 4.12. *Let $\gamma : K^* \rightarrow K$ be a non-degenerate simplicial dynamical system on an everywhere d -dimensional polyhedron X with g be the associated p.l. map on X . Let $J : {}^dK^* \rightarrow {}^dK$ be the inclusion map and $G^* = J^{-1} \circ \gamma$ the associated relation on the finite set ${}^dK^*$. With v_0 a positive distribution on dK , let $\lambda_0 = \sum_{s \in {}^dK} v_0(s) \lambda_s$ be the locally Lebesgue background measure on X .*

Let B^ and B be associated terminal basic sets for $({}^dK, G)$ and $({}^dK^*, G^*)$. On $H_\gamma(B_{G^*}^*)$ $\lambda_B = \sum_{s \in B} v_B(s) \lambda_s$ is an ergodic measure for g with support $X(\bar{B}^*) = X(\bar{B})$*

With the measures $\{\lambda_B\}$ indexed by the terminal basic sets of $({}^dK, G)$, the system (X, g) is tractable.

Proof: By Theorem 4.11, H_γ is a μ_0 almost one-to-one surjection. From Corollary 3.9 it follows that $({}^dK_{G^*}^*, S)$ is tractable and from Theorem 2.7 it then follows that the image system under H_γ is tractable. From Theorem 4.11 it follows that

$$(H_\gamma)_*(\sum_{s \in B} v_B(s) \mu_s) = \sum_{s \in B} v_B(s) \lambda_s = \lambda_B.$$

□

All of the locally Lebesgue measures obtained by taking different positive distributions v_0 on K are absolutely equivalent to one another and so any one can be used as the background measure. Clearly, for each terminal G basic set B the measure λ_B is absolutely continuous

with respect to λ_0 . In fact, it is a locally Lebesgue measure on the the d -dimensional polyhedron $X(\bar{B}) = X(\bar{B}^*)$.

It can happen that for distinct terminal basic sets B_1, B_2 the polyhedra $X(\bar{B}_1)$ and $X(\bar{B}_2)$ have a non-empty intersection, but only at boundary points, i.e. the intersection is a polyhedron of dimension less than d . In that case, and in other cases as well, $X(\bar{B}_1)$ and $X(\bar{B}_2)$ are contained in the same basic set for g . That is, the map which takes the G^* basic set B^* in ${}^dK^*$ to the g basic set which contains $X(\bar{B}^*)$ need not be injective. By Proposition 3.1 every terminal basic set of ${}^dK^*$ is equal to $B_{G^*}^*$ for some terminal G^* basic set and this has image $X(\bar{B}^*) = X(\bar{B})$. From Proposition 2.2 again it follows that every terminal g basic set contains $X(\bar{B})$ for some terminal G basic set B . A g basic set which contains $X(\bar{B})$ for a terminal G basic set $X(\bar{B})$ is necessarily visible. This was part of the proof for Theorem 2.7 of the TRAC 3 property. However, such a g basic set need not be itself terminal.

Examples (a) Let X be the interval $[0, 2]$ triangulated by K with vertices $\{0, 1, 2\}$. Let K' be the derived subdivision with additional vertices at $\frac{1}{2}$ and $\frac{3}{2}$. Define $\gamma : K' \rightarrow K$ by

$$(4.22) \quad 0, 1, 2 \mapsto 1, \quad \text{and} \quad \frac{1}{2} \mapsto 0, \frac{3}{2} \mapsto 2.$$

On ${}^1K = \{[0, 1], [1, 2]\}$, $G = ([0, 1], [0, 1]), ([1, 2], [1, 2])$. The map g is chain transitive and so X is the unique g basic set, which is, of course, terminal. $B_1 = \{[0, 1]\}$, $B_2 = \{[1, 2]\}$ are distinct terminal G basic sets with associated ergodic measures Lebesgue restricted to each interval.

(b) Let X be the interval $[0, 3]$ triangulated by K with vertices $\{0, 1, 2, 3\}$. Let K' be the derived subdivision with additional vertices at $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. Define $\gamma : K' \rightarrow K$ by

$$(4.23) \quad \begin{aligned} 0, 1 &\mapsto 1, & \text{and} & \quad \frac{1}{2} \mapsto 0, \frac{3}{2} \mapsto 2, \\ 2, 3 &\mapsto 3, & \text{and} & \quad \frac{5}{2} \mapsto 2. \end{aligned}$$

On 1K , $G = ([0, 1], [0, 1]), ([1, 2], [1, 2]), ([1, 2], [2, 3]), ([2, 3], [2, 3])$. The terminal G basic sets are $B_1 = \{[0, 1]\}$, and $B_2 = \{[2, 3]\}$. The g basic sets are $[0, 1]$ and $[2, 3]$. The latter is terminal. The g basic set $[0, 1]$ is visible but not terminal.

If f itself is a non-degenerate p.l. map we can use a special approximation.

Lemma 4.13. *Let L be a subcomplex of K . If L_1 is a subdivision of L then there exists a subdivision K_1 of K which agrees with L_1 on $|L|$.*

Proof: List the simplices $\{s_1, s_2, \dots\}$ of $K \setminus (L \cup V(K))$ so that $\dim s_i \leq \dim s_j$ when $i < j$. Let $K_1 = L_1 \cup V(K)$ on $L \cup V(K)$. Successively star at each simplex on the list, introducing a vertex $v(s_i) \in s_i^\circ$ and using the cone triangulation $v(s_i) * [K_1|_{\partial s_i}]$ to extend K_1 over s_i . \square

Proposition 4.14. *Let L_1 be a subdivision of a complex L . There exists a subdivision L_2 of L_1 and a non-degenerate simplicial map $\xi : L_2 \rightarrow L$ which is subordinate to the identity on L . That is, with h the p.l. map associated with ξ , $h(s) = s$ for each simplex $s \in L$.*

Proof: We proceed by induction on the dimension of L .

If the dimension is zero then $L_1 = L = V(L)$ and we use $L_2 = L_1$ and $\xi = 1_{V(L)}$.

Notice that if L is a one-simplex then there is a non-degenerate simplicial map from L_1 to L which fixes the endpoints iff there are an even number of vertices. So if L_1 has an odd number of vertices we obtain L_2 by introducing one additional vertex.

Now assume that the dimension of L is $n \geq 1$. Let \tilde{L} be the $n - 1$ skeleton and \tilde{L}_1 the subdivision of \tilde{L} which is the restriction of L_1 . By induction hypothesis there is a subdivision \tilde{L}_2 of \tilde{L}_1 and a non-degenerate simplicial map $\tilde{\xi} : \tilde{L}_2 \rightarrow \tilde{L}$ which is subordinate to the identity.

For each simplex $s \in {}^n L$, $\tilde{\xi}$ maps $\tilde{L}_2|_{\partial s} \rightarrow \tilde{L}|_{\partial s}$. $\tilde{L}_2|_{\partial s}$ is a subdivision of $\tilde{L}_1|_{\partial s}$ and by Lemma 4.13 we can extend the subdivision to a subdivision $\hat{L}_2|_s$ of $\tilde{L}_1|_{\partial s}$. Extend $\tilde{\xi}$ to $\hat{\xi}$ from \hat{L}_2 to L by mapping each vertex of $\hat{L}_2|_s \setminus \hat{L}_2|_{\partial s}$ to a vertex of s . The simplicial map $\hat{\xi}$ is non-degenerate on the ∂s . Apply Proposition 9.2 of [2] to obtain a subdivision $L_2|_s$ of $\hat{L}_2|_s$ which agrees with $\hat{L}_2|_{\partial s}$ on ∂s and non-degenerate simplicial $\xi : L_2|_s \rightarrow L|_s$ which agrees with $\hat{\xi}$ on $\hat{L}_2|_{\partial s}$. This defines $\xi : L_2 \rightarrow L$ subordinate to the identity on L . \square

Theorem 4.15. *Let L and K be triangulations of a polyhedron X with L_1 a common subdivision. If $\phi : L \rightarrow K$ is a non-degenerate simplicial map, then there exists K^* a proper subdivision of L_1 and a non-degenerate simplicial map $\xi : K^* \rightarrow L$ which is subordinate to the identity on L . The composition $\gamma = \phi \circ \xi : K^* \rightarrow K$ is a non-degenerate simplicial dynamical system.*

Proof: The derived subdivision L'_1 is a proper subdivision of L_1 . By Proposition 4.14 there exists K^* a subdivision of L'_1 and a non-degenerate simplicial map $\xi : K^* \rightarrow L$ which is subordinate to the identity on L . Since K^* is a subdivision of L'_1 it is a proper subdivision of L_1 and so is a proper subdivision of K . It follows that γ is a simplicial dynamical system. As the composition of non-degenerate simplicial maps, it is non-degenerate.

□

5. SHIFT-LIKE DYNAMICAL SYSTEMS

With A a finite set which we regard as an alphabet, let $X = A^{\mathbb{Z}_+}$ be the Cantor set of infinite sequences in A . On it is defined the surjective shift map \hat{S} by $\hat{S}(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}_+$. We use the notation \hat{S} because we will be using another shift map below.

For $[n] = \{0, \dots, n-1\}$, $A^{[n]}$ is the set of words of length n for any $n \in \mathbb{Z}_+$. We write $n(w)$ for the length of a word w . Define the projection $J_n : X \rightarrow A^{[n]}$ by $J_n(x)_i = x_i$ for $i \in [n]$. That is, $J_n(x)$ is the initial word of length n in x . Similarly, if $m \geq n$ we define $J_n : A^{[m]} \rightarrow A^{[n]}$.

As before, see (3.1), we use the metric d given by

$$(5.1) \quad d(x, y) = \inf \{1/(k+1) : x_i = y_i \text{ for all } i < k\}.$$

Thus, $J_n(x) = J_n(y)$ iff $d(x, y) \leq 1/(n+1)$.

We concatenate words in the obvious way. If s is a word and $x \in X$ then $sx \in X$ with

$$(5.2) \quad (sx)_i = \begin{cases} s_i & \text{for } i < n(s), \\ x_{i-n(s)} & \text{for } i \geq n(s). \end{cases}$$

If t is another word, we similarly define st to obtain a word of length $n(s) + n(t)$.

Definition 5.1. *Given positive integers n, k , we say that a continuous map f on X is $(n, n+k)$ continuous if $J_n(f(x))$ depends only on $J_{n+k}(x)$, or, equivalently, the relation $\gamma = J_n \circ f \circ (J_{n+k})^{-1} : A^{[n+k]} \rightarrow A^{[n]}$ is a mapping. In that case, we say that γ is associated with f .*

*Given a map $\gamma : A^{[n+k]} \rightarrow A^{[n]}$, we define the associated shift-like map g on X by $g(s^*x) = \gamma(s^*)x$ for $s^* \in A^{[n+k]}$. Thus, g is defined by*

$$(5.3) \quad \hat{S}^n \circ g = \hat{S}^{n+k}, \quad \text{and} \quad J_n \circ g = \gamma \circ J_{n+k},$$

Proposition 5.2. *If f is a continuous map on X then for any positive integer n there exists a positive integer k so that f is $(n, n+k)$ continuous.*

Proof: The continuous map f is uniformly continuous and so there exists k so that $d(x, y) \leq \frac{1}{n+k+1}$ implies $d(f(x), f(y)) \leq \frac{1}{n+1}$. From the definition (5.1) of the metric this says that $J_{n+k}(x) = J_{n+k}(y)$ implies $J_n(f(x)) = J_n(f(y))$. \square

Remark: It can happen that f is (n, n) continuous, i.e. we could use $k = 0$ for n . For example, this is true if f is the identity map. However, we will always choose $k > 0$.

For fixed positive integers n, k , let $K = A^{[n]}$, $K^* = A^{[n+k]}$ and $\bar{K} = A^{[k]}$ be the sets of words of length $n, n+k$ and k , respectively. Let $J = J_n : K^* \rightarrow K$ so that $s = J(s^*)$ is the initial word of length n and $\bar{J} : K^* \rightarrow \bar{K}$ so that $\bar{s} = \bar{J}(s^*)$ is the terminal word of length k . That is, $s^* = s\bar{s}$.

The set of cylinders $\mathcal{A}(K) = \{\langle s \rangle : s \in K\}$ is a clopen partition of X with $y \in \langle s \rangle$ iff $y = sx$ with $x = \hat{S}^n(y)$. The clopen partition $\mathcal{A}(K^*) = \{\langle s^* \rangle : s^* \in K^*\}$ is a refinement of $\mathcal{A}(K)$ with $\langle s^* \rangle \subset \langle s \rangle$ for $(s^*, s) \in K^* \times K$ iff $s = J(s^*)$. That is, we can regard J as the inclusion map from $\mathcal{A}(K^*)$ to $\mathcal{A}(K)$.

Given a map $\gamma : K^* \rightarrow K$ we obtain a special two-alphabet model with relations $G = \gamma \circ J^{-1}$ and $G^* = J^{-1} \circ \gamma$ on K and K^* , respectively. We can identify $(K^*)^{\mathbb{Z}_+}$ with $X = A^{\mathbb{Z}_+}$ via the homeomorphism which associates to $\mathbf{s}^* \in (K^*)^{\mathbb{Z}_+}$ the obvious infinite concatenation $s_0^* s_1^* \dots$ of successive words of length $n+k$. This identifies the shift S on $(K^*)^{\mathbb{Z}_+}$ with \hat{S}^{n+k} on X . As before, we restrict the shift S to the closed invariant subspace $K_{G^*}^*$.

We define the map $H_\gamma : K_{G^*}^* \rightarrow X$.

Let $\mathbf{s}^* = s_0^* s_1^* \dots \in K_{G^*}^*$. Let $s_j = J(s_j^*)$ and $\bar{s}_j = \bar{J}(s_j^*)$. Thus, $s_j^* = s_j \bar{s}_j$ and $\gamma(s_j^*) = s_{j+1}$ for all j . Define

$$(5.4) \quad H_\gamma(\mathbf{s}^*) = s_0 \bar{s}_0 \bar{s}_1 \bar{s}_2 \dots = s_0^* \bar{s}_1 \bar{s}_2 \dots$$

It follows that $g(H_\gamma(\mathbf{s}^*)) = s_1 \bar{s}_1 \bar{s}_2 \dots = H_\gamma(S(\mathbf{s}^*))$. That is, H_γ maps S to g . Furthermore,

$$(5.5) \quad J_{n+k}(H_\gamma(\mathbf{s}^*)) = s_0^* = p_0(\mathbf{s}^*),$$

with $p_0 : (K^*)^{\mathbb{Z}_+} \rightarrow K^*$ the first coordinate projection.

Theorem 5.3. *With $K = A^{[n]}$ and $K^* = A^{[n+k]}$, let $\gamma : K^* \rightarrow K$ be a map.*

- (a) *Assume f is a continuous map on X which is $(n, n+k)$ continuous and is associated with γ . Let $R^f : X \rightarrow K_{G^*}^*$ be defined by $R^f(x)_j = J_{n+k}(f^j(x))$. The map R^f is continuous and maps f on X to the shift S on $K_{G^*}^*$.*
- (b) *If g on X is the shift-like map associated to γ , then g is $(n, n+k)$ continuous with $\gamma = J_n \circ g \circ (J_{n+k})^{-1}$, i.e. γ is associated with g . The map R^g is a homeomorphism with inverse H_γ and so is a conjugacy from g on X to S on $K_{G^*}^*$.*
- (c) *If f_1 is a continuous map on X which is also associated with γ then*

$$(5.6) \quad \begin{aligned} J_n \circ f_1 &= \gamma \circ J_{n+k} = J_n \circ f. \\ \text{In particular, } J_n \circ f &= J_n \circ g, \end{aligned}$$

and so $d(f(x), g(x)) \leq \frac{1}{n+1}$ for all $x \in X$.

- (d) *The map $Q^f = (R^g)^{-1} \circ R^f = H_\gamma \circ R^f$ on X maps f to g . If $y = Q^f(x)$ then for every $j \in \mathbb{Z}_+$,*

$$(5.7) \quad \begin{aligned} J_{n+k}(f^j(x)) &= J_{n+k}(g^j(y)), \\ \text{and so } d(f^j(x), g^j(y)) &\leq \frac{1}{n+k+1} \text{ for all } j. \end{aligned}$$

Proof: (a) Clearly, $J \circ J_{n+k} = J_n : X \rightarrow K$ and so for any x and j ,

$$(5.8) \quad J(R^f(x)_{j+1}) = J_n(f^{j+1}(x)) = \gamma(J_{n+k}(f^j(x))) = \gamma(R^f(x)_j).$$

That is, $(R^f(x)_j, R^f(x)_{j+1}) \in G^*$ and so R^f maps X into $K_{G^*}^*$. Since $(K^*)^{\mathbb{Z}_+}$ is a product of finite discrete spaces, R^f is clearly continuous and clearly maps f to S .

(b) Since $J_n \circ g = \gamma \circ J_{n+k}$ and J_{n+k} is onto, it follows that $\gamma = J_n \circ g \circ (J_{n+k})^{-1}$.

We will show that H_γ is the inverse of R^g . This implies that the continuous map R^g is a bijection and so is a homeomorphism by compactness.

Let $\mathbf{s}^* \in K_{G^*}^*$. By (5.5) and (5.4)

$$(5.9) \quad s_j^* = (S^j(\mathbf{s}^*))_0 = J_{n+k}(H_\gamma(S^j(\mathbf{s}^*))) = J_{n+k}(g^j(H_\gamma(\mathbf{s}^*))) = R^g(H_\gamma(\mathbf{s}^*))_j.$$

That is, $\mathbf{s}^* = R^g(H_\gamma(\mathbf{s}^*))$, i.e. $R^g \circ H_\gamma = 1_{K_{G^*}^*}$.

Now let $x \in X$. Write $x = s_0 \bar{s}_0 \bar{s}_1 \dots$. That is, $s_0 = J_n(x)$ and $\bar{s}_j = x_{n+jk} \dots x_{n+(j+1)k-1}$. Define $s_j = J_n(g^j(x))$. By induction and the definition of g it follows that $g^j(x) = s_j \bar{s}_j \bar{s}_{j+1} \dots$ and $s_{j+1} = \gamma(s_j \bar{s}_j)$. Hence, $J_{n+k}(g^j(x)) = s_j \bar{s}_j$. Thus, $\bar{s}_j = \bar{J}(R^g(x)_j)$. Finally, $s_0 = J(R^g(x)_0)$. It follows that $H_\gamma(R^g(x)) = x$, i.e. $H_\gamma \circ R^g = 1_X$.

(c) Equation (5.6) is the definition of the statement that both f and f_1 are associated with γ .

(d) Equation (5.7) just says that $R^g \circ Q^f = R^f$.

The metric estimates in (c) and (d) follow from (5.1).

□

Thus, the shift-like system (X, g) is conjugate to the finite type subshift $(K_{G^*}^*, S)$. For each $x \in X$, $Q^f(x) \in X$ is a point whose g orbit $\frac{1}{n+k+1}$ shadows the f orbit of x .

Let N be the cardinality of the alphabet A (recall that $X = A^{\mathbb{Z}_+}$). On X we will use as the background measure λ_0 the $\frac{1}{N}$ Bernoulli measure. That is, $\lambda_0(\langle w \rangle) = \frac{1}{N^{n(w)}}$ where $n(w)$ is the length of the word. So if the word $w = s_0 \bar{s}_0 \dots \bar{s}_{p-1}$ with $s_0 \in K$ and $\bar{s}_0, \dots, \bar{s}_{p-1} \in \bar{K}$ then $\lambda_0(\langle w \rangle) = \frac{1}{N^{n+p k}}$. For $s \in K$ we let λ_s be the induced probability measure with support $\langle s \rangle$. So

$$(5.10) \quad \lambda_s(\langle s \bar{s}_0 \dots \bar{s}_{p-1} \rangle) = \frac{1}{N^{p k}}.$$

Clearly,

$$(5.11) \quad \lambda_0 = \sum_{s \in K} \lambda_0(s) \lambda_s = \frac{1}{N^n} \sum_{s \in K} \lambda_s.$$

For the special two alphabet model associated with $\gamma : K^* \rightarrow K$ we define the distribution data $\nu : K^* \rightarrow [0, 1]$ by

$$(5.12) \quad \nu(s^*) = \frac{1}{N^k} = \lambda_s(\langle \bar{s} \rangle) \quad \text{with } s^* = s \bar{s}.$$

By (3.16) the measure μ_s on $K_{G^*}^*$, is given by

$$(5.13) \quad \mu_s(\langle s_0^* \dots s_{p-1}^* \rangle) = \begin{cases} \frac{1}{N^{p k}} & \text{if } J(s_0^*) = s, \\ 0 & \text{otherwise,} \end{cases}$$

for $s_0^* \dots s_{p-1}^*$ a $K_{G^*}^*$ sequence.

Lemma 5.4. $(H_\gamma)_*(\mu_s) = \lambda_s$.

Proof: From (5.4) it is clear that for the $K_{G^*}^*$ sequence $s_0^* \dots s_{p-1}^*$ with $J(s_0^*) = s$ the homeomorphism H_γ maps $\langle s_0^* \dots s_{p-1}^* \rangle$ clopen in $K_{G^*}^*$ to $\langle s \bar{s}_0 \dots \bar{s}_{p-1} \rangle$ with $\bar{s}_j = \bar{J}(s_j^*)$ for $0 \leq j < p$. By (5.10) and (5.13) the measures agree.

□

As (B^*, B) varies over the pairs of associated basic sets for (K^*, G^*) and (K, G) , the subsets $B_{G^*}^*$ vary over the S basic sets in $K_{G^*}^*$. Let

$B_g^* = H_\gamma(B_{G^*}^*) \subset X$. Since H_γ is a conjugacy, these are the g basic sets. Furthermore, B_g^* is terminal iff B^* and B are.

If B is terminal then by (3.13) and (5.12) the stationary distribution v_B on B satisfies for $s_2 \in K$:

$$(5.14) \quad \frac{1}{N^k} \cdot \sum_{s^* \in B^* \cap \gamma^{-1}(s_2)} v_B(J(s^*)) = v_B(s_2).$$

Theorem 5.5. *Let A be a finite alphabet of cardinality N . Let g on $X = A^{\mathbb{Z}^+}$ be the shift-like map associated with $\gamma : K^* \rightarrow K$ for $K = A^{[n]}$ and $K^* = A^{[n+k]}$. Let $J = J_n : K^* \rightarrow K$ be the initial word map and $G^* = J^{-1} \circ \gamma$ be the associated relation on the finite set K^* . Let v_0 be the positive distribution on K with $v_0(s) = \lambda_0(s) = \frac{1}{N^n}$ for $s \in K$. Let $\lambda_0 = \sum_{s \in K} v_0(s) \lambda_s$, which is the $\frac{1}{N}$ Bernoulli measure, be the background measure on X .*

Let B^ and B be associated terminal basic sets for (K, G) and (K^*, G^*) . The measure $\lambda_B = \sum_{s \in B} v_B(s) \lambda_s$ is an ergodic measure for g with support $B_g^* = H_\gamma(B_G^*)$.*

With the measures $\{\lambda_B\}$ indexed by the terminal basic sets of (K, G) , the system (X, g) is tractable.

Proof: Since H_γ is a conjugacy which preserves the various measures, the result follows from Corollary 3.9.

□

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